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# AMERICAN
# Journal of Mathematics

EDITED BY

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WITH THE COÖPERATION OF

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AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

*Πραγμάτων ἔλεγχος οὐ βλεπομένων*

VOLUME XLI, NUMBER 2

BALTIMORE: THE JOHNS HOPKINS PRESS

LEMCKE & BUECHNER, *New York.* E. STEIGER & CO., *New York.* WILLIAM WESLEY & SON, *London.*
G. E. STECHERT & CO., *New York.* ARTHUR F. BIRD, *London.* A. HERMANN, *Paris.*

APRIL, 1919

Entered as Second-Class Matter at the Baltimore, Maryland, Postoffice.
Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917.
Authorized on July 3, 1918.

# *Asymptotic Satellites near the Straight-Line Equilibrium Points in the Problem of Three Bodies.*

By Daniel Buchanan.

## §1. Oscillating Satellites.

It was shown by Lagrange in a prize memoir* in 1772 that if two finite spheres revolve in circles about their common centre of mass, then there are three points on the line joining their centres at which an infinitesimal body would remain if it were given an initial projection so as to be instantaneously fixed with respect to the moving bodies. These points are called the *straight-line equilibrium points* of the problem of three bodies. Starting from minus infinity the order of the equilibrium points and the finite bodies is an equilibrium point, a finite body, a second equilibrium point, the other finite body, the third equilibrium point.

If the infinitesimal body is given an initial displacement from a point of equilibrium and initial conditions are so chosen that it moves in a closed orbit relatively to the moving system, it is then called an *oscillating satellite.*

The problem of the oscillating satellite has been discussed extensively. In the papers cited below,† the differential equations are limited to their linear terms and the orbits restricted to the plane of motion of the finite bodies.

A rigorous demonstration for the existence of periodic orbits for the oscillating satellite and a practical method for constructing them are given by Moulton in Chapter V of his "Periodic Orbits."‡ In this memoir the differential equations are unrestricted in the number of terms which may be taken,

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*Lagrange, "Collected Works," Vol. VI, pp. 229–324.

†The following references to the literature of the oscillating satellite are taken from Chapter V of Moulton's "Periodic Orbits":

Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Vol. I (1892), p. 159.

Burrau, *Astronomische Nachrichten*, Nos. 3230, 3251 (1894).

Perchot and Mascart, *Bulletin Astronomique*, Vol. XII (1895), p. 329. Moulton adds, "apparently their work is vitiated by an error in establishing the existence of the solutions, and their construction fails where they stopped."

Sir George H. Darwin, *Acta Mathematica*, Vol. XXI (1897), p. 99.

Plummer, *Monthly Notices, Royal Astronomical Society*, Vol. LXIII (1903), p. 436, and Vol. LXIV (1903), p. 98.

‡This memoir will be cited as the "Oscillating Satellite." Another method is given in Chapter VI of the "Periodic Orbits," but we are concerned with the former method only.

and the orbits are not limited to two dimensions as in the previous literature. Three classes of orbits are shown to exist and they are designated according to their periods as Class A, Class B, and Class C. The orbits of Class A and Class C are of three dimensions, while those of Class B are of two dimensions. The orbits of Class C are shown to exist under special conditions which might never be realized in the problem. Their period is a multiple of the periods of Class A and Class B when these latter periods are commensurable. Practical constructions are made for the orbits of Class A and of Class B, but, owing to the complexity of the problem, no attempt has been made to determine whether orbits of Class C exist which are distinct from those of Class A and Class B.

## § 2. Asymptotic Satellites.

The object of this paper is to obtain solutions of the differential equations of motion of the infinitesimal body which will approach the periodic solutions of Class A and Class B as the time approaches plus infinity or minus infinity. When the infinitesimal body moves in an orbit defined by such solutions, it will be called an *asymptotic satellite.*

The question of the existence of solutions which are asymptotic to the orbits of Class C is not considered in this paper, owing to the fact that it has not been determined in the "Oscillating Satellite" that orbits of Class C exist which are distinct from those of Class A and Class B.

The orbits which are asymptotic to the equilibrium points themselves have been determined by Warren.* These orbits are of two dimensions and lie in the plane of motion of the finite bodies.

The form of the asymptotic solutions is that adopted by Poincaré,† each term being of the type $e^{\lambda t}P(t)$, where $\lambda$ is a constant having its real part different from zero, and $P(t)$ is a constant or periodic function of $t$. It has been shown by Poincaré that solutions of this type will converge for all values of $t$, provided that certain divisors which appear in the construction of such solutions are different from zero.‡ If these divisors vanish, terms of the form $te^{\lambda t}P(t)$ will arise. If, therefore, the construction can be made so that no terms occur in $t$ explicitly, the divisors previously mentioned are different from zero and the solutions will converge for all values of $t$. Hence it is sufficient

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* Warren, "A Class of Asymptotic Orbits in the Problem of Three Bodies," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVIII, No. 3 (1916), pp. 221–248.

† Poincaré, "Mécanique Celéste," Vol. I, p. 340.

‡ Poincaré, *loc. cit.* p. 341.

to consider only the formal construction of the asymptotic solutions, and if solutions can be constructed so as to contain no terms in $t$ explicitly, their convergence is assured by Poincaré's theorem.

## § 3. The Differential Equations of Motion.

Let the motion of the infinitesimal body be referred to a set of rotating rectangular coordinates $\xi\eta\zeta$ of which the origin is at the centre of mass of the finite bodies, the $\xi$-axis is the line joining the finite bodies, and the $\xi\eta$-plane is the plane of their motion. The $\xi$- and $\eta$-axes rotate about the $\zeta$-axis in the direction of the motion of the finite bodies and with the same angular velocity. The units of length, mass, and time will be taken so that the distance between the finite bodies, the sum of their masses, and the Gaussian constant respectively shall each be unity. With the units thus chosen, the mean angular motion of the system is likewise unity. Let the masses of the finite bodies be denoted by $1-\mu$ and $\mu$, $0<\mu\leqq\frac{1}{2}$. On denoting the coordinates of the infinitesimal body by $\xi$, $\eta$, $\zeta$, and differentiation with respect to $t$ by primes, the differential equations of motion are *

$$\left.\begin{aligned} &\xi''-2\eta'=\frac{\partial U}{\partial \xi},\quad \eta''+2\xi'=\frac{\partial U}{\partial \eta},\quad \zeta''=\frac{\partial U}{\partial \zeta},\\ &2U=\xi^2+\eta^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}\\ &\quad=(1-\mu)\left(r_1^2+\frac{2}{r_1}\right)+\mu\left(r_2^2+\frac{2}{r_2}\right)-\zeta^2-\mu(1-\mu),\\ &r_1=\sqrt{(\xi+\mu)^2+\eta^2+\zeta^2},\quad r_2=\sqrt{(\xi-1+\mu)^2+\eta^2+\zeta^2}. \end{aligned}\right\} \tag{1}$$

The points of equilibrium are the solutions of the equations †

$$\frac{\partial U}{\partial \xi}=\frac{\partial U}{\partial \eta}=\frac{\partial U}{\partial \zeta}=0.$$

There are two sets of points which satisfy these equations. One set consists of the two points which are at the vertices of the two equilateral triangles on the opposite sides of the line joining the finite bodies. The orbits which are asymptotic to these points and also to the periodic oscillations near these points are discussed in another paper.‡ The other set of points consists of

* Moulton, "Introduction to Celestial Mechanics" (1914), p. 279.

† Moulton, "Introduction to Celestial Mechanics," p. 290; Charlier, "Die Mechanik des Himmels," Vol. II, pp. 102–111.

‡ This paper is now under consideration for publication.

three points which lie on the straight line joining the finite bodies. Let the coordinates of these points be denoted by $\xi_0, 0, 0$, where the particular value of $\xi_0$ depends upon the equilibrium point in question.* The points themselves will be denoted by $(a)$, $(b)$, and $(c)$, where $(a)$ lies between $+\infty$ and the finite mass $\mu$, $(b)$ between $\mu$ and $1-\mu$, and $(c)$ between $1-\mu$ and $-\infty$.

If the infinitesimal body is given a small displacement from an equilibrium point and a small velocity with respect to the finite masses such that

$$\left.\begin{aligned} \xi &= \xi_0 + \bar{x}, \quad \eta = 0 + \bar{y}, \quad \zeta = 0 + \bar{z}, \\ \xi' &= 0 + \bar{x}', \quad \eta' = 0 + \bar{y}', \quad \zeta = 0 + \bar{z}', \end{aligned}\right\} \tag{2}$$

then the differential equations (1) become†

$$\left.\begin{aligned} \bar{x}'' - 2\bar{y}' &= \frac{\partial U}{\partial \bar{x}} = +X(\bar{x}, \bar{y}^2, \bar{z}^2), \\ \bar{y}'' + 2\bar{x}' &= \frac{\partial U}{\partial \bar{y}} = \bar{y} Y(\bar{x}, \bar{y}^2, \bar{z}^2), \\ \bar{z}'' &= \frac{\partial U}{\partial \bar{z}} = \bar{z} Z(\bar{x}, \bar{y}^2, \bar{z}^2), \\ U &= \tfrac{1}{2}(1-\mu)\left(r_1^2 + \frac{2}{r_1}\right) + \tfrac{1}{2}\mu\left(r_2^2 + \frac{2}{r_2}\right) - \tfrac{1}{2}\bar{z}^2 - \tfrac{1}{2}\mu(1-\mu), \\ r_1 &= \sqrt{(\xi_0 + \bar{x} + \mu)^2 + \bar{y}^2 + \bar{z}^2}, \quad r_2 = \sqrt{(\xi_0 - 1 + \bar{x} + \mu)^2 + \bar{y}^2 + \bar{z}^2}, \end{aligned}\right\} \tag{3}$$

where $X$, $Y$, and $Z$ are power series in $\bar{x}$, $\bar{y}^2$, $\bar{z}^2$. These series converge within certain regions about the equilibrium points.‡

## § 4. The Periodic Orbits.

In showing the existence of periodic solutions of the differential equations of the "Oscillating Satellite" which correspond to equations (3), and later in making the construction of these solutions, the transformations

$$\bar{x} = \varepsilon x, \quad \bar{y} = \varepsilon y, \quad \bar{z} = \varepsilon z, \quad t - t_0 = (1+\delta)\tau \tag{4}$$

are made, where $\varepsilon$ is an arbitrary parameter and $\delta$ is determined as a function of $\varepsilon$ so that the solutions in $x$, $y$, and $z$ shall be periodic with the periodic $2\pi$ in $\tau$. On denoting differentiation with respect to $\tau$ by a dot over the variables, the differential equations (3) become as a consequence of (4)§

$$\left.\begin{aligned} \ddot{x} - 2(1+\delta)y &= (1+\delta)^2[X_1 + X_2\varepsilon + \ldots + X_k\varepsilon^{k-1} + \ldots], \\ \ddot{y} + 2(1+\delta)\dot{x} &= (1+\delta)^2[Y_1 + Y_2\varepsilon + \ldots + Y_k\varepsilon^{k-1} + \ldots], \\ \ddot{z} &= (1+\delta)^2[Z_1 + Z_2\varepsilon + \ldots + Z_k\varepsilon^{k-1} + \ldots], \end{aligned}\right\} \tag{5}$$

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* "Oscillating Satellite," equations (4).
† "Oscillating Satellite," equations (6).
‡ "Oscillating Satellite," § 77.
§ "Oscillating Satellite," equations (11).

where $X_k$, $Y_k$, and $Z_k$ are homogeneous polynomials of degree $k$ in $x$, $y$, and $z$. From (3) it is obvious that the $X_k$ are even in $y$ and $z$, $Y_k$ odd in $y$ and even in $z$, and $Z_k$ even in $y$ and odd in $z$. The explicit values of these terms up to $k=3$ are *

$$X_1=(1+2A)x, \quad X_2=\tfrac{3}{2}B(-2x^2+y^2+z^2), \quad X_3=2C(2x^3-3xy^2-3xz^2),$$
$$Y_1=(1-A)y, \quad Y_2=3Bxy, \quad Y_3=\tfrac{3}{2}C(-4x^2y+y^3+yz^2),$$
$$Z_1=-Az, \quad Z_2=3Bxz, \quad Z_3=\tfrac{3}{2}C(-4x^2z+y^2z+z^3),$$
$$A=\frac{1-\mu}{r_1^{(0)3}}+\frac{\mu}{r_2^{(0)3}}, \quad r_1^{(0)}=+\sqrt{(\xi_0+\mu)^2},$$
$$B=\begin{matrix}+\\-\end{matrix}\frac{1-\mu}{r_1^{(0)4}}\begin{matrix}+\\-\end{matrix}\frac{\mu}{r_2^{(0)4}}, \quad r_2^{(0)}=+\sqrt{(\xi_0-1+\mu)^2},$$
$$C=\frac{1-\mu}{r_1^{(0)5}}+\frac{\mu}{r_2^{(0)5}}.$$

The upper, middle, or lower signs are to be taken in $B$ according as the equilibrium point is $(a)$, $(b)$, or $(c)$ respectively.

If periodic solutions of (5) exist, their periods are determined from the periods of the solutions of the linear terms of (5). The solutions of these linear terms are †

$$\left.\begin{aligned} &x=K_1e^{i\sigma\tau}+K_2e^{-i\sigma\tau}+K_3e^{\rho\tau}+K_4e^{-\rho\tau}, \quad i=\sqrt{-1},\\ &y=in(K_1e^{i\sigma\tau}-K_2e^{-i\sigma\tau})+m(K_3e^{\rho\tau}-K_4e^{-\rho\tau}),\\ &z=K_5\cos\sqrt{A}\tau+K_6\sin\sqrt{A}\tau, \quad n=\frac{\sigma^2+1+2A}{2\sigma}, \quad m=\frac{\rho^2-1-2A}{2\rho}, \end{aligned}\right\} \tag{6}$$

where $K_1, \ldots, K_6$ are the constants of integration, and $\sigma^2$ and $\rho^2$ are the negative and positive roots respectively of the quadratic in $\lambda^2$,

$$\lambda^4+(2-A)\lambda^2+(1-A)(1+2A)=0. \tag{7}$$

There are three real periods in these solutions, viz., $2\pi/\sqrt{A}$, $2\pi/\sigma$, and $P=2j\pi/\sqrt{A}=2k\pi/\sigma$, where $j$ and $k$ are positive integers and $\sqrt{A}$ and $\sigma$ are commensurable.‡ These are the periods of the orbits of Class A, Class B, and Class C respectively.

*Orbits of Class A.*

The periodic solutions of Class A are §

$$\left.\begin{aligned} &\bar{x}_1=\varepsilon x_1=0\varepsilon+(a_1+b_1\cos 2\sqrt{A}\tau)\varepsilon^2+0\varepsilon^3+\ldots,\\ &\bar{y}_1=\varepsilon y_1=0\varepsilon+(c_1\sin 2\sqrt{A}\tau)\varepsilon^2+0\varepsilon^3+\ldots,\\ &\bar{z}_1=\varepsilon z_1=\left(\frac{1}{\sqrt{A}}\sin\sqrt{A}\tau\right)\varepsilon+0\varepsilon^2+d_1(3\sin\sqrt{A}\tau-\sin 3\sqrt{A}\tau)\varepsilon^3+\ldots,\\ &\delta=\delta_1=0\varepsilon+\delta_1^{(2)}\varepsilon^2+\ldots+\delta_1^{(2j)}\varepsilon^{2j}, \end{aligned}\right\} \tag{8}$$

* "Oscillating Satellite," equations (53), (15), (35).

† "Oscillating Satellite," equations (16), (27), and (28).

‡ In §83 of the "Oscillating Satellite" it is shown that there are infinitely many values of $\mu$ between 0 and $\frac{1}{2}$ such that $\sqrt{A}$ and $\sigma$ are commensurable.

§ "Oscillating Satellite," equations (71).

where
$$a_1=\frac{-3B}{4A(1+2A)}, \qquad b_1=\frac{3B(1+3A)}{4A(1-7A+18A^2)},$$
$$c_1=\frac{-3B}{\sqrt{A}(1-7A+18A^2)}, \quad d_1=\frac{3}{64A^{5/2}}\left[\frac{3B^2(1+3A)}{1-7A+18A^2}-C\right],$$
$$\delta_1^{(2)}=-\frac{9}{16A^2}\left[\frac{3B^2(1-3A+14A^2)}{(1+2A)(1-7A+18A^2)}-C\right].$$

The expressions for $\bar{x}_1$ and $\bar{y}_1$ are power series in $\varepsilon^2$ with sums of cosines and sines respectively of even multiples of $\sqrt{A}\tau$ in the coefficients. The highest multiple of $\sqrt{A}\tau$ which appears in the coefficient of $\varepsilon^k$ in each series is $k$. The solution for $\bar{z}_1$ is a power series in odd powers of $\varepsilon$ having sums of sines of odd multiples of $\sqrt{A}\tau$ in the coefficients. The highest multiple of $\sqrt{A}\tau$ which occurs in the coefficient of $\varepsilon^k$ is likewise $k$. The initial conditions are so chosen in the construction of these solutions that

$$\bar{z}(0)=0, \quad \dot{\bar{z}}(0)=\varepsilon,$$

that is, the parameter $\varepsilon$ is proportional to the initial projection of the infinitesimal body from the plane of motion. The actual projection is $\varepsilon/(1+\delta_1)$.

Numerical examples of these orbits have been considered in the "Oscillating Satellite" and the following results have been obtained when the ratio of the finite masses is ten to one, or $1-\mu=10/11$, $\mu=1/11$.*

TABLE 1.

| Coefficient. | Point (*a*). | Point (*b*). | Point (*c*). |
|---|---|---|---|
| $A$ | 2.548 | 6.510 | 1.082 |
| $\sigma^2$ | 2.811 | 6.820 | 1.144 |
| $\rho^2$ | 3.359 | 11.330 | 0.226 |
| $n$ | 2.657 | 3.990 | 2.014 |
| $m$ | −0.747 | − 0.397 | −3.091 |
| $B$ | 6.548 | −10.961 | −1.136 |
| $C$ | 18.283 | 55.740 | 1.196 |
| $a_1$ | −0.316 | 0.090 | 0.249 |
| $b_1$ | 0.151 | − 0.036 | −0.230 |
| $c_1$ | −0.112 | 0.018 | 0.226 |
| $d_1$ | −0.037 | − 0.020 | −0.002 |
| $\delta_1^{(2)}$ | 0.184 | 0.467 | 0.001 |

* This was the ratio used by Darwin in the memoir cited at the beginning of this article. The same ratio was adopted by Moulton in the "Oscillating Satellites" so that he might compare his results with those of Darwin.

*Orbits of Class B.*

The periodic solutions of Class B are *

$$\left.\begin{aligned} \bar{x}_2 &= \varepsilon x_2 = (\cos\sigma\tau)\varepsilon + (a_2 + b_2\cos\sigma\tau + c_2\cos 2\sigma\tau)\varepsilon^2 + \ldots, \\ \bar{y}_2 &= \varepsilon y_2 = (-n\sin\sigma\tau)\varepsilon + (-nb_2\sin\sigma\tau + d_2\sin 2\sigma\tau)\varepsilon^2 + \ldots, \\ \delta &= \delta_2 = 0\varepsilon + \delta_2^{(2)}\varepsilon^2 + \delta_2^{(3)}\varepsilon^3 + \ldots, \end{aligned}\right\} \quad (9)$$

where $a_2, b_2, c_2, d_2, \delta_2^{(2)}, \ldots$, are known constants. The algebraic expressions for these constants may be found in equations (101), (103), and (107) of the "Oscillating Satellite." Their particular values for $1-\mu=10/11$, $\mu=1/11$ are to be found in Table 2. The first seven coefficients which appear in Table 1 have been used in determining the values in Table 2.

TABLE 2.

| Coefficient. | Point (*a*). | Point (*b*). | Point (*c*). |
|---|---|---|---|
| $a_2$ | −4.078 | 8.172 | 0.554 |
| $b_2$ | 1.996 | − 4.976 | −0.038 |
| $c_2$ | 2.082 | − 3.196 | −0.516 |
| $nb_2$ | 5.305 | −19.855 | −0.077 |
| $d_2$ | 1.250 | − 1.478 | −0.276 |
| $\delta_2^{(2)}$ | 3.955 | 8.553 | −1.407 |

The expressions for $\bar{x}_2$ and $\bar{y}_2$ in (9) are power series in $\varepsilon$ with sums of cosines and sines respectively of $\sigma\tau$ in the coefficients. The highest multiple of $\sigma\tau$ which occurs in the coefficient of $\varepsilon^k$ is $k$. The initial conditions are so chosen for the construction of these solutions that

$$\bar{x}_2(0) = \varepsilon, \quad \bar{y}_2(0) = 0.$$

In this case the parameter $\varepsilon$ denotes the initial displacement along the $\xi$-axis of the infinitesimal body from an equilibrium point.

## § 5. The Equations of Variation.

Suppose now that the initial conditions for the periodic orbits are changed slightly. Then the infinitesimal body will deviate from its periodic orbit, and the amount and character of the deviation will depend upon the initial conditions chosen. The equations for the disturbed orbit, in terms of the variables $x$, $y$, and $z$, will be

$$x = x_j + p_j, \quad y = y_j + q_j, \quad z = z_j + r_j\dagger, \quad (10)$$

* "Oscillating Satellite," equations (97), (105).

† The $r_1$ and $r_2$ in this equation and in subsequent equations are different from $r_1$ and $r_2$ in equations (1) and (3).

where $\varepsilon x_j$, $\varepsilon y_j$, and $\varepsilon z_j$ represent the periodic orbits of Class A or Class B according as $j=1$ or 2 respectively. The additional terms $p_j$, $q_j$, and $r_j$ are functions of $\tau$ which depend upon the initial disturbance from the periodic orbit. We desire to determine these functions so that they shall have the asymptotic form already referred to in § 2.

On substituting (10) in (5), the differential equations which define $p_j$, $q_j$, and $r_j$ are found to be

$$\left.\begin{aligned} \ddot{p}_j-2(1+\delta_j)\dot{q}_j+(1+\delta_j)^2(P_{j1}p_j+P_{j2}q_j+P_{j3}r_j)&=(1+\delta_j)^2P_j,\\ \ddot{q}_j+2(1+\delta_j)\dot{p}_j+(1+\delta_j)^2(Q_{j1}p_j+Q_{j2}q_j+Q_{j3}r_j)&=(1+\delta_j)^2Q_j,\\ \ddot{r}_j+(1+\delta_j)^2(R_{j1}p_j+R_{j2}q_j+R_{j3}r_j)&=(1+\delta_j)^2R_j, \end{aligned}\right\} \tag{11}$$

where

$$\begin{aligned} P_{j1}&=-1-2A+6B\varepsilon x_j-6C\varepsilon^2(2x_j^2-y_j^2-z_j^2)+\ldots.,\\ P_{j2}&=Q_{j1}=-3B\varepsilon y_j+12C\varepsilon^2x_jy_j+\ldots.,\\ P_{j3}&=R_{j1}=-3B\varepsilon z_j+12C\varepsilon^2x_jz_j+\ldots.,\\ Q_{j2}&=-1+A-3B\varepsilon x_j+\tfrac{3}{2}C\varepsilon^2(4x_j^2-3y_j^2-z_j^2)+\ldots.,\\ Q_{j3}&=R_{j2}=-3C\varepsilon^2y_jz_j+\ldots.,\\ R_{j3}&=A-3B\varepsilon x_j+\tfrac{3}{2}C\varepsilon^2(4x_j^2-y_j^2-3z_j^2)+\ldots.,\\ P_j&=\tfrac{3}{2}B\varepsilon(-2p_j^2+q_j^2+r_j^2)+()\varepsilon^2+\ldots.,\\ Q_j&=3B\varepsilon p_jq_j+()\varepsilon^2+\ldots.,\\ R_j&=3B\varepsilon p_jr_j+()\varepsilon^2+\ldots.. \end{aligned}$$

As we propose later to construct numerical solutions which are asymptotic to the particular periodic orbits mentioned above, we insert here the particular values of the various $P_{jk}$, $Q_{jk}$, and $R_{jk}$ which are obtained by making use of the numerical values listed in Tables 1 and 2. The following results are found.

*Orbits of Class A.*

*Equilibrium point (a).*

$$\begin{aligned} P_{11}&=-6.096+\varepsilon^2(8.384-15.686\cos 2\sqrt{A}\tau)+()\varepsilon^4+\ldots.,\\ P_{12}&=Q_{21}=\varepsilon^2(2.200\sin 2\sqrt{A}\tau)+()\varepsilon^4+\ldots.,\\ P_{13}&=R_{11}=-\varepsilon(12.297\sin\sqrt{A}\tau)+()\varepsilon^3+\ldots.,\\ Q_{12}&=1.548+\varepsilon^2(0.833+2.409\cos 2\sqrt{A}\tau)+()\varepsilon^4+\ldots.,\\ Q_{13}&=R_{12}=()\varepsilon^3+\ldots.\\ R_{13}&=2.548-\varepsilon^2(9.918-13.160\cos 2\sqrt{A}\tau)+()\varepsilon^4+\ldots.. \end{aligned}$$

*Equilibrium point (b).*

$$P_{11}=-14.020+\varepsilon^2(19.777-23.328\cos 2\sqrt{A}\tau)+()\varepsilon^4+\ldots.,$$
$$P_{12}=Q_{21}=\varepsilon^2(0.592\sin 2\sqrt{A}\tau)+()\varepsilon^4+\ldots.,$$
$$P_{13}=R_{11}=\varepsilon(12.890\sin\sqrt{A}\tau)+()\varepsilon^3+\ldots.,$$
$$Q_{12}=5.510+\varepsilon^2(-3.464+5.240\cos 2\sqrt{A}\tau)+()\varepsilon^4+\ldots.,$$
$$Q_{13}=R_{12}=()\varepsilon^3+\ldots.,$$
$$R_{13}=6.510+\varepsilon^2(-16.312+18.108\cos 2\sqrt{A}\tau)+()\varepsilon^4+\ldots..$$

*Equilibrium point (c).*

$$P_{11}=-3.164+\varepsilon^2(1.616-1.746\cos 2\sqrt{A}\tau)+()\varepsilon^4+\ldots.,$$
$$P_{12}=Q_{21}=\varepsilon^2(0.770\sin 2\sqrt{A}\tau)+()\varepsilon^4+\ldots.,$$
$$P_{13}=R_{11}=\varepsilon(3.275\sin\sqrt{A}\tau)+()\varepsilon^3+\ldots.,$$
$$Q_{12}=0.082+\varepsilon^2(0.020+0.044\cos 2\sqrt{A}\tau)+()\varepsilon^4+\ldots.,$$
$$Q_{13}=R_{12}=()\varepsilon^3+\ldots.,$$
$$R_{13}=1.082+\varepsilon^2(0.020+0.044\cos 2\sqrt{A}\tau)+\ldots..$$

*Orbits of Class B.*

*Equilibrium point (a).*

$$P_{21}=-6.10+\varepsilon(39.30\cos\sigma\tau)+\ldots., \quad P_{22}=Q_{21}=\varepsilon(52.27\sin\sigma\tau)+\ldots.,$$
$$Q_{22}=1.55-\varepsilon(19.64\cos\sigma\tau)+\ldots., \quad P_{23}=Q_{23}=R_{21}=R_{22}=0,$$
$$R_{23}=2.55-\varepsilon(19.64\cos\sigma\tau)+\ldots..$$

*Equilibrium point (b).*

$$P_{21}=-14.02-\varepsilon(65.77\cos\sigma\tau)+\ldots., \quad P_{22}=Q_{21}=-\varepsilon(131.20\sin\sigma\tau)+\ldots.,$$
$$Q_{22}=5.51+\varepsilon(32.88\cos\sigma\tau)+\ldots., \quad P_{23}=Q_{23}=R_{21}=R_{22}=0,$$
$$R_{23}=6.51+\varepsilon(32.88\cos\sigma\tau)+\ldots..$$

*Equilibrium point (c).*

$$P_{21}=-3.16-\varepsilon(6.82\cos\sigma\tau)+\ldots., \quad P_{22}=Q_{21}=\varepsilon(6.86\sin\sigma\tau)+\ldots.,$$
$$Q_{22}=0.08+\varepsilon(3.41\cos\sigma\tau)+\ldots., \quad P_{23}=Q_{23}=R_{21}=R_{22}=0,$$
$$R_{23}=1.08+\varepsilon(3.41\cos\sigma\tau)+\ldots..$$

If we consider only the terms of (11) which are linear in $p_j$, $q_j$, and $r_j$, we obtain the *equations of variation.** For the orbits of Class A they are

$$\left.\begin{aligned}\ddot{p}_1-2(1+\delta_1)\dot{q}_1+(1+\delta_1)^2(P_{11}p_1+P_{12}q_1+P_{13}r_1)&=0,\\ \ddot{q}_1+2(1+\delta_1)\dot{p}_1+(1+\delta_1)^2(Q_{11}p_1+Q_{12}q_1+Q_{13}r_1)&=0,\\ \ddot{r}_1+(1+\delta_1)^2(R_{11}p_1+R_{12}q_1+R_{13}r_1)&=0,\end{aligned}\right\} \qquad (12A)$$

and for orbits of Class B, they are

$$\left.\begin{aligned}\ddot{p}_2-2(1+\delta_2)\dot{q}_2+(1+\delta_2)^2(P_{21}p_2+P_{22}q_2)&=0,\\ \ddot{q}_2+2(1+\delta_2)\dot{p}_2+(1+\delta_2)^2(Q_{21}p_2+Q_{22}q_2)&=0,\\ \ddot{r}_2+(1+\delta_2)^2R_{23}r_2&=0.\end{aligned}\right\} \qquad (12B)$$

* Poincaré, *loc. cit.*, Vol. I, Chap. 4.

The periodic solutions (8) and (9) are called the *generating solutions** of (12A) and (12B) respectively.

## § 6. The Solutions of the Equations of Variation (12A).

The equations of variation are linear differential equations having periodic coefficients, the periods being the same as those of the corresponding generating solutions. Differential equations of this type were first treated by Hill† in his celebrated memoir on the lunar theory. A very extensive list of references to the literature of these differential equations was given by Baker in a recent memoir. ‡

The method which we shall adopt in the construction of the solutions of these equations is the one developed by Moulton and Macmillan.§ The form‖ of the solution is, in general

$$e^{\alpha\tau}\phi(\tau),$$

where $\phi(\tau)$ is a periodic function having the same period as the coefficients of the differential equations, and $\alpha$ is a constant called the *characteristic exponent.*

It has been shown by Poincaré¶ that if the generating solutions contain an arbitrary constant which does not appear explicitly in the original differential equations, then a set of solutions of the equations of variation is obtained by taking the first partial derivatives of the generating solutions with respect to this constant. The characteristic exponent associated with this set of solutions is zero, and the solutions themselves are either periodic or consist of periodic functions plus $\tau$ times other periodic functions. The initial time $t_0$ is one such arbitrary constant which is present in all differential equations of dynamics in which the potential function *e.g.*, $U$ in equations (1), does not contain the time explicitly. Differentiation of the generating solutions with respect to this constant yields a set of periodic solutions. Another such constant usually present is the scale factor of the generating solutions. Differentiation of the generating solutions with respect to this constant yields a set of solutions comprised of a periodic function plus $K\tau$, $K$ a constant, times the periodic function obtained by differentiating with respect to $t_0$.

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* Poincaré, *loc. cit.* Vol. I, Chap. 4.

† G. W. Hill, "Collected Works," Vol. I, p. 243; *Acta Mathematica*, Vol. VIII, pp. 1–36.

‡ H. F. Baker, "On Certain Linear Differential Equations of Astronomical Interest," *Philosophical Transactions of the Royal Society of London*, Series A (1916), Vol. 216, pp. 129–186.

§ Moulton and Macmillan, "On the Solutions of Certain Types of Linear Differential Equations with Periodic Coefficients," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIII (1911), No. 1, pp. 63–97.

‖ Floquet, *Annales Scientifiques de l'École Superieure*, Series II, Vol. XII (1883), p. 47.

¶ Poincaré, *loc cit.*, Vol. I Chap. 4.

Let us consider first equations (12A), and those solutions which have characteristic exponents different from zero. These equations are simultaneous and must be considered together. To find their solutions let

$$p_1=K_1e^{\alpha_1\tau}u_1, \quad q_1=L_1e^{\alpha_1\tau}v_1, \quad r_1=M_1e^{\alpha_1\tau}w_1, \tag{13}$$

where $K_1$, $L_1$, and $M_1$ are arbitrary constants. An existence proof, which is omitted here, would show that

$$\left.\begin{aligned} \alpha_1 &= \alpha_1^{(0)}+\alpha_1^{(2)}\varepsilon^2+\ldots.+\alpha_1^{(2k)}\varepsilon^{2k}+\ldots., \\ u_1 &= u_1^{(0)}+u_1^{(2)}\varepsilon^2+\ldots.+u_1^{(2k)}\varepsilon^{2k}+\ldots., \\ v_1 &= v_1^{(0)}+v_1^{(2)}\varepsilon^2+\ldots.+v_1^{(2k)}\varepsilon^{2k}+\ldots., \\ w_1 &= w_1^{(1)}\varepsilon+w_1^{(3)}\varepsilon^3+\ldots.+w_1^{(2k+1)}\varepsilon^{2k+1}+\ldots., \end{aligned}\right\} \tag{14}$$

where the $\alpha_1^{(2k)}$ are constants so determined that the various $u_1^{(2k)}$, $v_1^{(2k)}$, and $w_1^{(2k+1)}$ shall be periodic with the period $2\pi/\sqrt{A}$ in $\tau$. On substituting (13) in (12A) and considering only the terms independent of $\varepsilon$, it is found that $\alpha_1^{(0)}$ must satisfy the biquadratic (7) if the arbitrary constants $K_1$, $L_1$, and $M_1$ are to have values other than the trivial ones $K_1=L_1=M_1=0$. It is shown in the "Oscillating Satellite," § 82, that this biquadratic admits two purely imaginary solutions, denoted by $+\sigma i$ and $-\sigma i$, and two real solutions, denoted by $+\rho$ and $-\rho$, for all values of $\mu$ such that $0\leqq\mu\leqq\frac{1}{2}$, and for each equilibrium point $(a)$, $(b)$, and $(c)$.

Since the functions $u_1$, $v_1$, and $w_1$ are multiplied by arbitrary constants, we may assume without loss of generality, that $u_1(0)=1$. Hence

$$u_1^{(0)}(0)=1, \quad u_1^{(2k)}(0)=0 \qquad (k=1, 2, \ldots.\infty). \tag{15}$$

Then $v_1(0)$ and $w_1(0)$ can be determined from the differential equations which define $v_1$ and $w_1$.

The various solutions of (12A) may be constructed by substituting (14) and (13) in (12A), and giving to $\alpha_1^{(0)}$ the values $\sigma i$, $-\sigma i$, $\rho$, and $-\rho$ in turn. By equating the coefficients of the various powers of $\varepsilon$ we obtain sets of differential equations which can be integrated step by step. The arbitrary constants of integration arising at each step can be uniquely determined from the conditions (15). The various $\alpha_1^{(2k)}$ can likewise be uniquely determined by the condition that $u_1^{(2k)}$, $v_1^{(2k)}$, and $w_1^{(2k+1)}$ shall be periodic with the period $2\pi/\sqrt{A}$ in $\tau$. From Poincaré's extension to Cauchy's theorem,* it is known that these solutions will converge for $\varepsilon$ sufficiently small numerically and for all values of $\tau$ such that $0\leqq\tau\leqq T$, where $T$ is an arbitrary period chosen in advance. Obviously, we may choose the period $2\pi/\sqrt{A}$.

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*Poincaré, *loc. cit.*, Vol. I, p. 58.

Let us denote the solutions of (12A) when $\alpha_1^{(0)}=\sigma i$ by

$$p_1=e^{i\sigma_1\tau}u_{11},\quad q_1=e^{i\sigma_1\tau}iv_{11},\quad r_1=e^{i\sigma_1\tau}iw_{11}. \tag{16}$$

The arbitrary constants are omitted here but they appear in equations (25). It is found that $\sigma_1$ is a power series in $\varepsilon^2$ with real constant coefficients. The functions $u_{11}$ and $v_{11}$ are likewise power series in $\varepsilon^2$, but their coefficients are sums of cosines and $i$ times sines of even multiples of $\sqrt{A}\tau$. The function $w_{11}$ is a power series in odd powers of $\varepsilon$, vanishing with $\varepsilon$, and the coefficients are sums of cosines and $i$ times sines of odd multiples of $\sqrt{A}\tau$. In each of these three functions, the highest multiple of $\sqrt{A}\tau$ which appears in the coefficient of $\varepsilon^k$ is $k$. So far as the computation has been carried out, we have

$$u_{11}=1+()\varepsilon^2+\ldots.,\quad v_{11}=n+()\varepsilon^2+\ldots.,$$

$$w_{11}=\frac{3B}{4A-\sigma^2}\left[\frac{2}{\sigma}\cos\sqrt{A}\tau-\frac{i}{\sqrt{A}}\sin\sqrt{A}\tau\right]\varepsilon+()\varepsilon^3+\ldots.,$$

$$\sigma_1=\sigma+\varepsilon^2\left[\frac{9B^2\{\sigma^2(1-13A)-(3+7A-22A^2)\}}{16A\sigma^3(4A-\sigma^2)}-\frac{\delta_1^{(2)}(1-A)(1+2A)}{\sigma^3}\right]+\ldots..$$

Since the differential equations (12A) do not contain $i$, they are unaltered by a change in the sign of $i$. If then we change the sign of $i$ in the differential equations and in the initial conditions (15), we obtain solutions which differ from (16) only in the sign of $i$. Let this set of solutions be denoted by

$$p_1=e^{-i\sigma\tau}u_{12},\quad q_1=-e^{-i\sigma\tau}iv_{12},\quad r_1=-e^{-i\sigma\tau}iw_{12}, \tag{17}$$

where $u_{12}$, $v_{12}$, and $w_{12}$ differ from $u_{11}$, $v_{11}$, and $w_{11}$ respectively only in the sign of $i$. The same result would be obtained by putting $\alpha_1^{(0)}=-i\sigma$ and proceeding as in the construction of the solution (16).

Likewise by putting $\alpha_1^{(0)}$ equal, first to $\rho$, and then to $-\rho$, we obtain the respective solutions

$$p_1=e^{\rho_1\tau}u_{13},\quad q_1=e^{\rho_1\tau}v_{13},\quad r_1=e^{\rho_1\tau}w_{13}, \tag{18}$$

and

$$p_1=e^{-\rho_1\tau}u_{14},\quad q_1=-e^{-\rho_1\tau}v_{14},\quad r_1=-e^{-\rho_1\tau}w_{14}, \tag{19}$$

where $\rho_1$ is a power series in $\varepsilon^2$ with real constant coefficients. Further $u_{13}$, $u_{14}$; $v_{13}$, $v_{14}$; $w_{13}$, $w_{14}$ are similar to $u_{11}$, $v_{11}$, and $w_{11}$ respectively, except that the coefficients of the former functions are all real; also

$$u_{14}(\tau)=u_{13}(-\tau),\quad v_{14}(\tau)=v_{13}(-\tau),\quad w_{14}(\tau)=w_{13}(-\tau).$$

The computation for the first terms of these series gives

$$u_{13}=1+()\varepsilon^2+\ldots.,\quad v_{13}=m+()\varepsilon^2+\ldots.,$$

$$w_{13}=-\frac{3B}{4A+\rho^2}\left[\frac{2}{\rho}\cos\sqrt{A}\tau-\frac{1}{\sqrt{A}}\sin\sqrt{A}\tau\right]\varepsilon+()\varepsilon^3+\ldots.,$$

$$\rho_1=\rho+\varepsilon^2\left[\frac{9B^2\{\rho^2(1-13A)+3+7A-22A^2\}}{14A\rho^2(4A+\rho^2)}+\frac{\delta_1^{(2)}(1-A)(1+2A)}{\rho^3}\right]+\ldots..$$

Obviously, the solutions (19) differ from (18) only in the sign of $\tau$, $q_1$, and $r_1$. This property can be derived directly from the differential equations, which we proceed to show. We shall show that if

$$p_j=F_{j1}(\tau), \quad q_j=F_{j2}(\tau), \quad r_j=F_{j3}(\tau) \tag{20}$$

are solutions of the differential equations (11), then

$$p_j=F_{j1}(-\tau), \quad q_j=-F_{j2}(-\tau), \quad r_j=-F_{j3}(-\tau)$$

are likewise solutions. If this property holds for equations (11) it will evidently hold for equations (12A) or (12B), *i. e.*, when the right members of (11) are omitted.

Let us consider the differential equations (1). Since $U$ is even in $\eta$ and $\zeta$, it follows that $\frac{\partial U}{\partial \xi}$ is even in $\eta$ and $\zeta$, $\frac{\partial U}{\partial \eta}$ is odd in $\eta$ and even in $\zeta$, and $\frac{\partial U}{\partial \zeta}$ is even in $\eta$ and odd in $\zeta$. After the substitutions (2) are made, the right members of (3) are functions of $\bar{y}$ and $\bar{z}$ as indicated. As the substitutions (4) do not alter the parity of $y$ and $z$, then it follows that the right members of (5) have the same parity in $y$ and $z$ that the corresponding right members have in $\bar{y}$ and $\bar{z}$. When the substitutions (10) are made in (5) we obtain (11), and the terms of (11) which carry $(1+\delta_j)^2$ as a factor in each equation arise from the corresponding right members of (5). Let these terms be denoted by (11, 1), (11, 2), and (11, 3) respectively. Then they have the following properties:

(11, 1) is even in $\{y_j, q_j\}$, and even in $\{z_j, r_j\}$,
(11, 2) is odd in $\{y_j, q_j\}$, and even in $\{z_j, r_j\}$,
(11, 3) is even in $\{y_j, q_j\}$, and odd in $\{z_j, r_j\}$,

where the braces $\{\}$ denote that the variables within are to be considered together. Further, since $\bar{y}$ is a factor of the right member of the second equation in (3), it follows that (11, 2) carries a factor which is odd in $\{y_j, q_j\}$. Similarly, since $\bar{z}$ is a factor of the right members of the third equation in (3), then (11, 3) carries a factor which is odd in $\{z_j, q_j\}$.

Now, if the signs of $y_j$, $z_j$, $q_j$, and $r_j$ be changed in the above expressions, then (11, 1) remains unchanged while (11, 2) and (11, 3) change signs. On examining the generating solutions (8) and (9) we observe that changing the signs of $y_j$ and $z_j$ is equivalent to changing the sign of $\tau$ in these solutions. Thus, if we change the signs of $\tau$, $q_j$, and $r_j$, (11, 1) remains unchanged while (11, 2) and (11, 3) change signs. If these changes are made in all the terms of (11), the first equation remains unchanged while each term of the remaining equations changes sign and the factor $-1$ can be divided out. Thus the equations (11) remain unchanged if we change the signs of $\tau$, $q_j$ and $r_j$.

Suppose now that (20) is a set of solutions of (11). Let

$$\bar{p}_j=p_j, \quad \bar{q}_j=-q_j, \quad \bar{r}_j=-r_j, \quad \bar{\tau}=-\tau. \tag{21}$$

Then the differential equations obtained by making these substitutions in (11) are identically the same in $\bar{p}_j$, $\bar{q}_j$, $\bar{r}_j$, and $\bar{\tau}$ as (11) are in the former variables. With the same initial conditions for $\bar{p}_j$, $\bar{q}_j$, $\bar{r}_j$, and $\bar{\tau}$ as for equations (11), we obtain the solutions

$$\bar{p}_j=F_{j1}(\bar{\tau}), \quad \bar{q}_j=F_{j2}(\bar{\tau}), \quad \bar{r}_j=F_{j3}(\bar{\tau}),$$

where $F_{j1}$, $F_{j2}$, and $F_{j3}$ are the same functions as in (20). On restoring the former variables by (21) we have

$$p_j=F_{j1}(-\tau), \quad q_j=-F_{j2}(-\tau), \quad r_j=-F_{j3}(-\tau)$$

as solutions of (11). Therefore if a set of solutions of (11) is known, another set can be obtained by changing the signs of $\tau$, $q_j$, and $r_j$ in the former set.

Let us next consider the two remaining sets of solutions of (12A). These solutions have characteristic exponents zero and may be obtained, as already referred to in this section, by differentiating the generating solutions (8) with respect to the arbitrary constants $t_0$ and $\varepsilon$. Taking the constant $t_0$ first, we have the solutions

$$p_1=\frac{\partial \bar{x}_1}{\partial t_0}=\frac{\partial \bar{x}_1}{\partial \tau}\frac{\partial \tau}{\partial t_0}=-\frac{1}{1+\delta_1}\left[(-2b_1\sqrt{A}\sin 2\sqrt{A}\tau)\varepsilon^2+()\varepsilon^4+\dots.\right]$$
$$=-\frac{1}{1+\delta_1}u_{15},$$
$$q_1=\frac{\partial \bar{y}_1}{\partial t_0}=\frac{\partial \bar{y}_1}{\partial \tau}\frac{\partial \tau}{\partial t_0}=-\frac{1}{1+\delta_1}\left[(2c_1\sqrt{A}\cos 2\sqrt{A}\tau)\varepsilon^2+()\varepsilon^4+\dots.\right]$$
$$=-\frac{1}{1+\delta_1}v_{15},$$
$$r_1=\frac{\partial \bar{z}_1}{\partial t_0}=\frac{\partial \bar{z}_1}{\partial \tau}\frac{\partial \tau}{\partial t_0}=-\frac{1}{1+\delta_1}\left[(\cos\sqrt{A}\tau)\varepsilon+()\varepsilon^3+\dots.\right]=-\frac{1}{1+\delta_1}w_{15}.$$

As these solutions are later multiplied by arbitrary constants, we may neglect the factor $-\dfrac{1}{1+\delta_1}$ and consider

$$p_1=u_{15}, \quad q_1=v_{15}, \quad r_1=w_{15} \tag{22}$$

as the solutions. Similarly, on differentiating (8) with respect to $\varepsilon$ we obtain

$$p_1=\frac{\partial \bar{x}_1}{\partial \varepsilon}, \quad q_1=\frac{\partial \bar{y}_1}{\partial \varepsilon}, \quad r_1=\frac{\partial \bar{z}_1}{\partial \varepsilon}.$$

Since $\tau=(t-t_0)/(1+\delta_1)$ and $\delta_1$ is a function of $\varepsilon$, then the constant $\varepsilon$ enters

the generating solutions not only explicitly but implicitly through $\delta_1$ and $\tau$. Hence

$$\frac{\partial \bar{x}_1}{\partial \varepsilon} = \left(\frac{\partial \bar{x}_1}{\partial \varepsilon}\right) + \frac{\partial \bar{x}_1}{\partial \tau}\frac{\partial \tau}{\partial \delta_1}\frac{\partial \delta_1}{\partial \varepsilon}, \quad \frac{\partial \bar{y}_1}{\partial \varepsilon} = \left(\frac{\partial \bar{y}_1}{\partial \varepsilon}\right) + \frac{\partial \bar{y}_1}{\partial \tau}\frac{\partial \tau}{\partial \delta_1}\frac{\partial \delta_1}{\partial \varepsilon},$$

$$\frac{\partial \bar{z}_1}{\partial \varepsilon} = \left(\frac{\partial \bar{z}_1}{\partial \varepsilon}\right) + \frac{\partial \bar{z}_1}{\partial \tau}\frac{\partial \tau}{\partial \delta_1}\frac{\partial \delta_1}{\partial \varepsilon},$$

where the parentheses around the partial derivatives denote that the differentiation is performed only in so far as $\varepsilon$ occurs explicitly. When the differentiations are performed, we have

$$p_1 = u_{16} + K\tau u_{15}, \quad q_1 = v_{16} + K\tau v_{15}, \quad r_1 = w_{16} + K\tau w_{15}, \tag{23}$$

where

$$u_{16} = \left(\frac{\partial \bar{x}_1}{\partial \varepsilon}\right) = 2(a_1 + b_1 \cos 2\sqrt{A}\tau)\varepsilon + ()\varepsilon^3 + \ldots.,$$

$$v_{16} = \left(\frac{\partial \bar{y}_1}{\partial \varepsilon}\right) = 2(c_1 \sin 2\sqrt{A}\tau)\varepsilon + ()\varepsilon^3 + \ldots.,$$

$$w_{16} = \left(\frac{\partial \bar{z}_1}{\partial \varepsilon}\right) = \frac{1}{\sqrt{A}} \sin \sqrt{A}\tau + ()\varepsilon^2 + \ldots.,$$

$$K = -\frac{1}{1+\delta_1}[2\delta_1^{(2)}\varepsilon + ()\varepsilon^3 + \ldots.].$$

It remains now to show that the solutions which we have obtained, viz., (16), (17), (18), (19), (22), and (23) constitute a fundamental set. The criterion for a fundamental set is that the determinant formed from these solutions and their first derivatives with respect to $\tau$ shall be different from zero. This determinant is

$$\Delta_1 = \begin{vmatrix}
e^{i\sigma_1\tau}u_{11}, & e^{-i\sigma_1\tau}u_{12}, & e^{\rho_1\tau}u_{13}, & e^{-\rho_1\tau}u_{14}, & u_{15}, u_{16}+K\tau u_{15}, \\
e^{i\sigma_1\tau}(i\sigma_1 u_{11}+\dot{u}_{11}), & e^{-i\sigma_1\tau}(-i\sigma_1 u_{12}+\dot{u}_{12}), & e^{\rho_1\tau}(\rho_1 u_{13}+\dot{u}_{13}), & e^{-\rho_1\tau}(-\rho_1 u_{14}+\dot{u}_{14}), & \dot{u}_{15}, \dot{u}_{16}+K(\tau\dot{u}_{15}+u_{15}), \\
e^{i\sigma_1\tau}iv_{11}, & e^{-i\sigma_1\tau}(-iv_{12}), & e^{\rho_1\tau}v_{13}, & e^{-\rho_1\tau}(-v_{14}), & v_{15}, v_{16}+K\tau v_{15}, \\
e^{i\sigma_1\tau}(-\sigma_1 v_{11}+i\dot{v}_{11}), & e^{-i\sigma_1\tau}(\sigma_1 v_{12}-i\dot{v}_{12}), & e^{\rho_1\tau}(\rho_1 v_{13}+\dot{v}_{13}), & e^{-\rho_1\tau}(\rho_1 v_{14}-\dot{v}_{14}), & \dot{v}_{15}, \dot{v}_{16}+K(\tau\dot{v}_{15}+v_{15}), \\
e^{i\sigma_1\tau}iw_{11}, & e^{-i\sigma_1\tau}(-iw_{12}), & e^{\rho_1\tau}w_{13}, & e^{-\rho_1\tau}(-w_{14}), & w_{15}, w_{16}+K\tau w_{15}, \\
e^{i\sigma_1\tau}(-\sigma_1 w_{11}+i\dot{w}_{11}), & e^{-i\sigma_1\tau}(\sigma_1 w_{12}-i\dot{w}_{12}), & e^{\rho_1\tau}(\rho_1 w_{13}+\dot{w}_{13}), & e^{-\rho_1\tau}(\rho_1 w_{14}-\dot{w}_{14}), & \dot{w}_{15}, \dot{w}_{16}+K(\tau\dot{w}_{15}+w_{15}).
\end{vmatrix} \tag{24}$$

It is a constant,* and its value can be determined with the least difficulty by putting $\tau = 0$. Thus we obtain

$$\Delta_1 = 4i\varepsilon(m\rho+n\sigma)(m\sigma-n\rho) + \text{ terms of higher degree in } \varepsilon.$$

It is shown in the "Oscillating Satellite," equations (36), that $m\rho+n\sigma$ and $m\sigma-n\rho$ are different from zero. Hence the determinant $\Delta_1$ is different from

* Moulton, "Periodic Orbits," Chap. 1, Sec. 18.

zero for $\varepsilon$ not zero, but sufficiently small numerically. Therefore the solutions which have been obtained for (12A) constitute a fundamental set, and the most general solutions are

$$\left.\begin{aligned} p_1 &= K_1 e^{i\sigma_1\tau} u_{11} + K_2 e^{-i\sigma_1\tau} u_{12} + K_3 e^{\rho_1\tau} u_{13} + K_4 e^{-\rho_1\tau} u_{14} + K_5 u_{15} \\ &\qquad + K_6(u_{16} + K\tau u_{15}), \\ q_1 &= K_1 e^{i\sigma_1\tau} i v_{11} - K_2 e^{-i\sigma_1\tau} i v_{12} + K_3 e^{\rho_1\tau} v_{13} - K_4 e^{-\rho_1\tau} v_{14} + K_5 v_{15} \\ &\qquad + K_6(v_{16} + K\tau v_{15}), \\ r_1 &= K_1 e^{i\sigma_1\tau} i w_{11} - K_2 e^{-i\sigma_1\tau} i w_{12} + K_3 e^{\rho_1\tau} w_{13} - K_4 e^{-\rho_1\tau} w_{14} + K_5 w_{15} \\ &\qquad + K_6(w_{16} + K\tau w_{15}), \end{aligned}\right\} \tag{25}$$

where $K_1, \ldots, K_6$ are arbitrary constants.

## § 7. The Solutions of the Equations of Variation (12B).

The solutions of the equations (12B) are obtained in the same way as the preceding solutions were found. The construction is simplified by the fact that the last equation is independent of the first two.

The two sets of solutions of the first two equations of (12B) which have characteristic exponents different from zero, are

$$p_2 = e^{\rho_2\tau} u_{21}, \quad q_2 = e^{\rho_2\tau} v_{21}, \tag{26}$$

and

$$p_2 = e^{-\rho_2\tau} u_{22}, \quad q_2 = -e^{-\rho_2\tau} v_{22}, \tag{27}$$

where $\rho_2$ is a power series in $\varepsilon$ with real coefficients, the terms of lower degrees being

$$\rho_2 = \rho + 0\varepsilon + \rho_2^{(2)}\varepsilon^2 + \rho_2^{(3)}\varepsilon^3 + \ldots.$$

The functions $u_{21}$, $v_{21}$, $u_{22}$, and $v_{22}$ are power series in $\varepsilon$ with sums of sines and cosines of multiples of $\sigma\tau$ in the coefficients, the highest multiple of $\sigma\tau$ which occurs in the coefficient of $\varepsilon^k$ being $k$. All the numerical coefficients in these series are real. From the property of the differential equations (11) which was proved in the preceding section, it follows that the solutions (27) are obtained from (26) by changing the signs of $\tau$ and $q_2$. Thus $u_{22}$ and $v_{22}$ differ from $u_{21}$ and $v_{21}$ respectively only in the signs of the sines.

Since these solutions (26) and (27) are multiplied later by arbitrary constants, we may take $u_{21}(0) = 1$. The algebraic forms of the functions $u_{21}$ and $v_{21}$ are

$$\begin{aligned} u_{21} &= 1 + (a_{21}\cos\sigma\tau + b_{21}\sin\sigma\tau)\varepsilon + ()\varepsilon^2 + \ldots., \\ v_{21} &= m + (a_{22}\cos\sigma\tau + b_{22}\sin\sigma\tau)\varepsilon + ()\varepsilon^2 + \ldots., \\ u_{22}(\tau) &= u_{21}(-\tau), \quad v_{22}(\tau) = v_{21}(-\tau). \end{aligned}$$

The two remaining solutions of the first two equations of (12B) are obtained by differentiating the generating solutions (9) with respect to $t_0$ and $\varepsilon$. Differentiation with respect to $t_0$ gives

$$p_2=\frac{\partial\bar{x}_2}{\partial t_0}=-\frac{1}{1+\delta_2}\frac{\partial\bar{x}_2}{\partial\tau},\quad q_2=\frac{\partial\bar{y}_2}{\partial t_0}=-\frac{1}{1+\delta_2}\frac{\partial\bar{y}_2}{\partial\tau},$$

and, since the multiplier $-\frac{1}{1+\delta_2}$ may be dropped, the solutions become

$$\left.\begin{aligned} p_2=u_{23}&=\frac{\partial\bar{x}_2}{\partial\tau}=(-\sigma\sin\sigma\tau)\varepsilon-(b_2\sigma\sin\sigma\tau+2c_2\sigma\sin 2\sigma\tau)\varepsilon^2+\ldots.,\\ q_2=v_{23}&=\frac{\partial\bar{y}_2}{\partial\tau}=(-n\sigma\cos\sigma\tau)\varepsilon-(nb_2\sigma\cos\sigma\tau-2d_2\sigma\cos 2\sigma\tau)\varepsilon^2+\ldots..\end{aligned}\right\}\quad(28)$$

Differentiation with respect to $\varepsilon$ gives

$$p_2=u_{24}+L\tau u_{23},\quad q_2=v_{24}+L\tau v_{23},\qquad(29)$$

where

$$u_{24}=\left(\frac{\partial\bar{x}_2}{\partial\varepsilon}\right)=\cos\sigma\tau+2(a_2+b_2\cos\sigma\tau+c_2\cos 2\sigma\tau)\varepsilon+\ldots.,$$

$$v_{24}=\left(\frac{\partial\bar{y}_2}{\partial\varepsilon}\right)=-n\sin\sigma\tau-2(nb_2\sin\sigma\tau-d_2\sin 2\sigma\tau)\varepsilon+\ldots.,$$

$$L=-\frac{1}{1+\delta_2}(2\delta_2^{(2)}\varepsilon+3\delta_2^{(3)}\varepsilon^2+\ldots.).$$

The determinant of these solutions (26), (27), (28), and (29) and their first derivatives with respect to $\tau$ is

$$\Delta_2=\begin{vmatrix} e^{\rho_2\tau}u_{21}, & e^{-\rho_2\tau}u_{22}, & u_{23}, & u_{24}+L\tau u_{23},\\ e^{\rho_2\tau}(\rho_2u_{21}+\dot{u}_{21}), & e^{-\rho_2\tau}(-\rho_2u_{22}+\dot{u}_{22}), & \dot{u}_{23}, & \dot{u}_{24}+L(\tau\dot{u}_{23}+u_{23}),\\ e^{\rho_2\tau}v_{21}, & e^{-\rho_2\tau}(-v_{22}), & v_{23}, & v_{24}+L\tau v_{23},\\ e^{\rho_2\tau}(\rho_2v_{21}+\dot{v}_{21}), & e^{-\rho_2\tau}(\rho_2v_{22}-\dot{v}_{22}), & \dot{v}_{23}, & \dot{v}_{24}+L(\tau\dot{v}_{23}+v_{23}),\end{vmatrix}\qquad(30)$$

$$=2\sigma\varepsilon(m\rho+n\sigma)(m\sigma-n\rho)+\text{terms of higher degree in }\varepsilon.$$

As in the former determinant $\Delta_1$, equation (24), the factors $m\rho+n\sigma$ and $m\sigma-n\rho$ are different from zero, and since $\sigma\neq 0$ it follows that $\Delta_2\neq 0$ for $\varepsilon$ not zero, but sufficiently small numerically. Hence the solutions (26), (27), (28), and (29) constitute a fundamental set of solutions of the first two equations, and the most general solutions are

$$\left.\begin{aligned} p_2&=L_1e^{\rho_2\tau}u_{21}+L_2e^{-\rho_2\tau}u_{22}+L_3u_{23}+L_4(u_{24}+L\tau u_{23}),\\ q_2&=L_1e^{\rho_2\tau}v_{21}-L_2e^{-\rho_2\tau}v_{22}+L_3v_{23}+L_4(v_{24}+L\tau v_{23}),\end{aligned}\right\}\qquad(31)$$

where $L_1,\ldots.,L_4$ are arbitrary constants.

The general solution of the last equation of (12B) is readily obtained, and it has the form

$$r_2 = L_5 e^{i\omega\tau} w_{21} + L_6 e^{-i\omega\tau} w_{22}, \tag{32}$$

where $L_5$ and $L_6$ are arbitrary constants, and $\omega$, $w_{21}$, and $w_{22}$ are power series in $\varepsilon$ of the form

$$w_{21} = 1 + \frac{3B\varepsilon}{\sigma^2(\sigma^2-4A)}\,(\sigma^2\cos\sigma\tau - 2i\sqrt{A}\sin\sigma\tau - \sigma^2) + \ldots.,$$

$$w_{22} = 1 + \frac{3B\varepsilon}{\sigma^2(\sigma^2-4A)}\,(\sigma^2\cos\sigma\tau + 2i\sqrt{A}\sin\sigma\tau - \sigma^2) + \ldots.,$$

$$\omega = \sqrt{A} + 0\varepsilon + \omega_2\varepsilon^2 + \ldots..$$

Since $L_5$ and $L_6$ are arbitrary, the initial values of $w_{21}$ and $w_{22}$ may be chosen so that $w_{21}(0) = w_{22}(0) = 1$. The coefficients of the various powers of $\varepsilon$ in these functions are sums of cosines and $i$ times sines of multiples of $\sigma\tau$, the highest multiple which occurs in the coefficient of $\varepsilon^k$ being $k$. Further, $w_{21}$ and $w_{22}$ differ only in the sign of $i$. In the series for $\omega$, all the coefficients are real. The determinant of the two solutions in (32) and their derivatives with respect to $\tau$ is

$$\Delta_3 = \begin{vmatrix} e^{i\omega\tau} w_{21}, & e^{-i\omega\tau} w_{22}, \\ e^{i\omega\tau}(i\omega w_{21} + \dot{w}_{21}), & e^{-i\omega\tau}(-i\omega w_{21} + \dot{w}_{21}), \end{vmatrix}$$
$$= -2i\omega + \text{terms in } \varepsilon,$$

and this is different from zero for $|\varepsilon|$ sufficiently small. Hence the solutions (32) constitute a fundamental set of solutions of the last equation of (12B).

## § 8. The Construction of Asymptotic Solutions.

Having determined the solutions of the equations of variation, we shall now proceed to construct the asymptotic solutions of (11). We shall consider first the solutions which approach zero as $\tau$ approaches $+\infty$, and then show how to obtain from these the solutions which approach zero as $\tau$ approaches $-\infty$.

In making the constructions, it is convenient to introduce a parameter $\gamma$ by the substitutions

$$p_j = \bar{p}_j\gamma, \quad q_j = \bar{q}_j\gamma, \quad r_j = \bar{r}_j\gamma, \tag{33}$$

where $\bar{p}_j$, $\bar{q}_j$, and $\bar{r}_j$ are new dependent variables. Then, since the asymptotic solutions are to be constructed as power series in $\gamma$, we put

$$\left.\begin{aligned} \bar{p}_j = p_j^{(0)} + p_j^{(1)}\gamma + p_j^{(2)}\gamma^2 + \ldots., \quad \bar{q}_j = q_j^{(0)} + q_j^{(1)}\gamma + q_j^{(2)}\gamma^2 + \ldots., \\ \bar{r}_j = r_j^{(0)} + r_j^{(1)}\gamma + r_j^{(2)}\gamma^2 + \ldots.. \end{aligned}\right\} \tag{34}$$

When (34) and (33) are substituted in (11), the factor $\gamma$ will divide out. Let the resulting differential equations be denoted by (11′). On equating the coefficients of the various powers of $\gamma$ in these equations, we obtain sets of differential equations which define the functions $p_j^{(k)}$, $q_j^{(k)}$, and $r_j^{(k)}$ in (34). In order that the solutions shall be asymptotic we must impose the condition

($C_1$), that each term of the solutions for the various $p_j^{(k)}$, $q_j^{(k)}$, and $r_j^{(k)}$ shall contain $e^{-\rho_j \tau}$ as a factor.

As an arbitrary constant arises at each step of the integration we may impose the further condition

($C_2$), that $p_j(0)=\alpha_j$,

from which it follows that

$$p_j^{(0)}(0)=\alpha_j, \quad p_j^{(k)}(0)=0, \quad (k=1, \ldots . \infty).$$

*Asymptotic Orbits of Class A.*

Let us consider first the orbits which are asymptotic to the periodic orbits of Class A. We put $j=1$ and consider the various differential equations obtained by equating the coefficients of the same powers of $\gamma$ in (11′).

For the terms in $p_1^{(0)}$, $q_1^{(0)}$, and $r_1^{(0)}$ we obtain a set of differential equations which are the same as (12A), except for the superscript 0. The general solutions are the same as (25), and when the condition ($C_1$) is imposed all the constants of integration must be zero except the one associated with $e^{-\rho_1 \tau}$. From condition ($C_2$) it follows that this constant must have the value $\alpha_1$. Therefore the desired solutions are

$$p_1^{(0)}=\alpha_1 e^{-\rho_1 \tau} u_{14}, \quad q_1^{(0)}=-\alpha_1 e^{-\rho_1 \tau} v_{14}, \quad r_1^{(0)}=-\alpha_1 e^{-\rho_1 \tau} w_{14}. \tag{35}$$

The differential equations which are obtained from equating the coefficients of $\gamma$ to the first degree in (11′) have the same left members as (12A), except for the superscript 1 on the variables. On denoting the corresponding right members by $P_1^{(1)}$, $Q_1^{(1)}$, and $R_1^{(1)}$, we find that

$$P_1^{(1)}=\varepsilon \alpha_1^2 e^{-2\rho_1 \tau} U_{12}^{(1)}, \quad Q_1^{(1)}=\varepsilon \alpha_1^2 e^{-2\rho_1 \tau} V_{12}^{(1)}, \quad R_1^{(1)}=\varepsilon \alpha_1^2 e^{-2\rho_1 \tau} W_{12}^{(1)},$$

where $U_{12}^{(1)}$, $V_{12}^{(1)}$, and $W_{12}^{(1)}$ are power series of the same form as $u_{14}$, $v_{14}$, and $w_{14}$ respectively. The complementary functions of these differential equations are the same as (25), but let us suppose that the arbitrary multipliers are

$k_1^{(1)}, \ldots, k_6^{(1)}$. The particular integrals can be obtained by varying these multipliers. Thus

$$\left.\begin{aligned}
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}u_{11}+\dot{k}_2^{(1)}e^{-i\sigma_1\tau}u_{12}+\dot{k}_3^{(1)}e^{\rho_1\tau}u_{13}+\dot{k}_4^{(1)}e^{-\rho_1\tau}u_{14}+\dot{k}_5^{(1)}u_{15}\\
&\qquad+\dot{k}_6^{(1)}(u_{16}+K\tau u_{15})=0,\\
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}(i\sigma_1u_{11}+\dot{u}_{11})-\dot{k}_2^{(1)}e^{-i\sigma_1\tau}(i\sigma_1u_{12}-\dot{u}_{12})+\dot{k}_3^{(1)}e^{\rho_1\tau}(\rho_1u_{13}+\dot{u}_{13})\\
&\qquad-\dot{k}_4^{(1)}e^{-\rho_1\tau}(\rho_1u_{14}-\dot{u}_{14})+\dot{k}_5^{(1)}\dot{u}_{15}+\dot{k}_6^{(1)}[\dot{u}_{16}+K(u_{15}+\tau\dot{u}_{15})]=P_1^{(1)},\\
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}iv_{11}-\dot{k}_2^{(1)}e^{-i\sigma_1\tau}iv_{12}+\dot{k}_3^{(1)}e^{\rho_1\tau}v_{13}-\dot{k}_4^{(1)}e^{-\rho_1\tau}v_{14}+\dot{k}_5^{(1)}v_{15}\\
&\qquad+\dot{k}_6^{(1)}(v_{16}+K\tau v_{15})=0,\\
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}(-\sigma_1v_{11}+i\dot{v}_{11})+\dot{k}_2^{(1)}e^{-i\sigma_1\tau}(\sigma_1v_{12}-i\dot{v}_{12})+\dot{k}_3^{(1)}e^{\rho_1\tau}(\rho_1v_{13}+\dot{v}_{13})\\
&\qquad+\dot{k}_4^{(1)}e^{-\rho_1\tau}(\rho_1v_{14}-\dot{v}_{14})+\dot{k}_5^{(1)}\dot{v}_{15}+\dot{k}_6^{(1)}[\dot{v}_{16}+K(v_{15}+\tau\dot{v}_{15})]=Q_1^{(1)},\\
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}iw_{11}-\dot{k}_2^{(1)}e^{-i\sigma_1\tau}iw_{12}+\dot{k}_3^{(1)}e^{\rho_1\tau}w_{13}-\dot{k}_4^{(1)}e^{-\rho_1\tau}w_{14}+\dot{k}_5^{(1)}w_{15}\\
&\qquad+\dot{k}_6^{(1)}(w_{16}+K\tau w_{15})=0,\\
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}(-\sigma_1w_{11}+i\dot{w}_{11})+\dot{k}_2^{(1)}e^{-i\sigma_1\tau}(\sigma_1w_{12}-i\dot{w}_{12})+\dot{k}_3^{(1)}e^{\rho_1\tau}(\rho_1w_{13}+\dot{w}_{13})\\
&\qquad+\dot{k}_4^{(1)}e^{-\rho_1\tau}(\rho_1w_{14}-\dot{w}_{14})+\dot{k}_5^{(1)}\dot{w}_{15}+\dot{k}_6^{(1)}[\dot{w}_{16}+K(w_{15}+\tau\dot{w}_{15})]=R_1^{(1)}.
\end{aligned}\right\}\quad(36)$$

The determinant of the coefficients of $\dot{k}_1^{(1)}, \ldots, \dot{k}_6^{(1)}$ is $\Delta_1$, equation (24), and since it is different from zero for $\varepsilon$ not zero but sufficiently small numerically these equations can be solved for $\dot{k}_1^{(1)}, \ldots, \dot{k}_6^{(1)}$. Thus

$$\dot{k}_l^{(1)}=\frac{\Delta_{1l}^{(1)}}{\Delta_1}, \qquad (l=1, \ldots, 6), \tag{37}$$

where $\Delta_{1l}^{(1)}$ is the determinant formed by replacing the elements of the $l$-th column of $\Delta_1$, with 0, $P_1^{(1)}$, 0, $Q_1^{(1)}$, 0, and $R_1^{(1)}$ respectively. Since $P_1^{(1)}$, $Q_1^{(1)}$, and $R_1^{(1)}$ contain no terms in $e^{\pm i\sigma_1\tau}$ or $e^{\pm\rho_1\tau}$, the integrations of (37) for $k_1^{(1)}, \ldots, k_4^{(1)}$ will yield no terms in $\tau$ explicitly. Such terms will occur, however, in the integrations for $k_5^{(1)}$ and $k_6^{(1)}$, but when they are substituted in the complementary functions they will cancel off. The complete solutions are thus found to be

$$\left.\begin{aligned}
p_1^{(1)}&=K_1^{(1)}e^{i\sigma_1\tau}u_{11}+K_2^{(1)}e^{-i\sigma_1\tau}u_{12}+K_3^{(1)}e^{\rho_1\tau}u_{13}+K_4^{(1)}e^{-\rho_1\tau}u_{14}+K_5^{(1)}u_{15}\\
&\qquad+K_6^{(1)}(u_{16}+K\tau u_{15})+\varepsilon\alpha_1^2e^{-2\rho_1\tau}u_{12}^{(1)},\\
q_1^{(1)}&=K_1^{(1)}e^{i\sigma_1\tau}iv_{11}-K_2^{(1)}e^{-i\sigma_1\tau}iv_{12}+K_3^{(1)}e^{\rho_1\tau}v_{13}-K_4^{(1)}e^{-\rho_1\tau}v_{14}+K_5^{(1)}v_{15}\\
&\qquad+K_6^{(1)}(v_{16}+K\tau v_{15})+\varepsilon\alpha_1^2e^{-2\rho_1\tau}v_{12}^{(1)},\\
r_1^{(1)}&=K_1^{(1)}e^{i\sigma_1\tau}iw_{11}-K_2^{(1)}e^{-i\sigma_1\tau}iw_{12}+K_3^{(1)}e^{\rho_1\tau}w_{13}-K_4^{(1)}e^{-\rho_1\tau}w_{14}+K_5^{(1)}w_{15}\\
&\qquad+K_6^{(1)}(w_{16}+K\tau w_{15})+\varepsilon\alpha_1^2e^{-2\rho_1\tau}w_{12}^{(1)},
\end{aligned}\right\}\quad(38)$$

where $K_1^{(1)}, \ldots, K_6^{(1)}$ are the constants of integration, and $u_{12}^{(1)}$, $v_{12}^{(1)}$, and $w_{12}^{(1)}$ are functions similar to $u_{14}$, $v_{14}$, and $w_{14}$ respectively. On imposing condition $(C_1)$ upon these solutions we obtain

$$K_1^{(1)}=K_2^{(1)}=K_3^{(1)}=K_5^{(1)}=K_6^{(1)}=0,$$

and from condition $(C_2)$ we have the remaining constant

$$K_4^{(1)} = -\varepsilon\alpha_1^2 u_{12}^{(1)}(0).$$

When the constants of integration are thus determined, the solutions (38) take the form

$$\left.\begin{aligned} p_1^{(1)} &= \varepsilon\alpha_1^2(e^{-\rho_1\tau}u_{11}^{(1)} + e^{-2\rho_1\tau}u_{12}^{(1)}), \quad q_1^{(1)} = \varepsilon\alpha_1^2(e^{-\rho_1\tau}v_{11}^{(1)} + e^{-2\rho_1\tau}v_{12}^{(1)}), \\ r_1^{(1)} &= \varepsilon\alpha_1^2(e^{-\rho_1\tau}w_{11}^{(1)} + e^{-2\rho_1\tau}w_{12}^{(1)}), \end{aligned}\right\} \quad (39)$$

where the functions $u_{11}^{(1)}$, $u_{12}^{(1)}$; $v_{11}^{(1)}$, $v_{12}^{(1)}$; $w_{11}^{(1)}$, $w_{12}^{(1)}$ are of the same form as $u_{14}$, $v_{14}$, and $w_{14}$ respectively.

The remaining steps of the integration can be carried on in essentially the same way. By an induction to the general term we shall show that the solutions of (11) can be constructed to any desired degree of accuracy.

Let us assume that $p_1^{(\nu)}$, $q_1^{(\nu)}$, $r_1^{(\nu)}$ have been constructed for $\nu = 1, 2, \ldots, k-1$, and that

$$p_1^{(\nu)} = \varepsilon^\nu\alpha_1^{\nu+1}\sum_{l=1}^{\nu+1} e^{-l\rho_1\tau}u_{1l}^{(\nu)}, \quad q_1^{(\nu)} = \varepsilon^\nu\alpha_1^{\nu+1}\sum_{l=1}^{\nu+1} e^{-l\rho_1\tau}v_{1l}^{(\nu)}, \quad r_1^{(\nu)} = \varepsilon^\nu\alpha_1^{\nu+1}\sum_{l=1}^{\nu+1} e^{-l\rho_1\tau}w_{1l}^{(\nu)}, \quad (40)$$

where the functions $u_{1l}^{(\nu)}$, $v_{1l}^{(\nu)}$, and $w_{1l}^{(\nu)}$ are similar to $u_{14}$, $v_{14}$, and $w_{14}$ respectively. We propose to show that the solutions for $p_1^{(k)}$, $q_1^{(k)}$, and $r_1^{(k)}$ are the same as (40) if $\nu = k$.

Consider the set of differential equations obtained by equating the coefficients of $\gamma^k$ in (11′). The left members are the same as (12A), except for the superscript $k$ on $p_1$, $q_1$, and $r_1$. Let the corresponding right members be denoted by $P_1^{(k)}$, $Q_1^{(k)}$, and $R_1^{(k)}$. Then it is found that

$$P_1^{(k)} = \varepsilon^k\alpha_1^{k+1}\sum_{l=1}^{k+1} e^{-l\rho_1\tau}U_{1l}^{(k)}, \quad Q_1^{(k)} = \varepsilon^k\alpha_1^{k+1}\sum_{l=1}^{k+1} e^{-l\rho_1\tau}V_{1l}^{(k)}, \quad R_1^{(k)} = \varepsilon^k\alpha_1^{k+1}\sum_{l=1}^{k+1} e^{-l\rho_1\tau}W_{1l}^{(k)},$$

where $U_{1l}^{(k)}$, $V_{1l}^{(k)}$, and $W_{1l}^{(k)}$ are functions of the same form as $u_{14}$, $v_{14}$, and $w_{14}$ respectively. The complementary functions of these equations are the same as at each preceding step of the integration, and the complete solutions may also be obtained in the same way as equations (38) were determined. The equations analogous to (36) and (37) differ from (36) and (37) respectively only in the superscript $k$ instead of 1. The complete solutions will therefore have the same form as (38), except that the respective particular integrals are functions similar to $P_1^{(k)}$, $Q_1^{(k)}$, and $R_1^{(k)}$ instead of $P_1^{(1)}$, $Q_1^{(1)}$, and $R_1^{(1)}$ as in (38). From condition $(C_1)$ the arbitrary constants will all be zero, except $K_4^{(k)}$, and from condition $(C_2)$ this constant is found to carry the factor $\varepsilon^k\alpha_1^{k+1}$. Hence the desired solutions for $p_1^{(k)}$, $q_1^{(k)}$, and $r_1^{(k)}$ are the same as (40) if $\nu = k$. This completes the induction.

When these solutions for $p_1^{(\nu)}$, $q_1^{(\nu)}$, and $r_1^{(\nu)}$, $(\nu=1, 2, \ldots \infty)$ are substituted in (34) and (33), we obtain as the solutions of (11),

$$\left.\begin{aligned} p_1 &= \sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{-k\rho_1\tau}\, u_{1k}^{(j)}(\tau)\,\varepsilon^{j-1}\alpha_1^j\gamma^j, \\ q_1 &= \sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{-k\rho_1\tau}\, v_{1k}^{(j)}(\tau)\,\varepsilon^{j-1}\alpha_1^j\gamma^j, \\ r_1 &= \sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{-k\rho_1\tau} w_{1k}^{(j)}(\tau)\,\varepsilon^{j-1}\alpha_1^j\gamma^j, \end{aligned}\right\} \tag{41}$$

where the superscript on the functions $u_{1k}^{(j)}(\tau)$, $v_{1k}^{(j)}(\tau)$, and $w_{1k}^{(j)}(\tau)$ has been made to conform with the corresponding power of $\gamma$. Since the two arbitrary parameters $\alpha_1$ and $\gamma$ occur only in products as indicated, we may suppress either without loss of generality. Let us suppose that $\alpha_1=1$.

Equations (41) are the asymptotic solutions of (11) which approach zero as $\tau$ and therefore $t$ approach $+\infty$. In the same way there could be obtained the asymptotic solutions which approach zero as $\tau$ and therefore $t$ approach $-\infty$. It is not necessary to construct these solutions, however, since they can be obtained directly from (41), as we proceed to show.

It was shown in the two paragraphs following equation (20) that if a set of solutions of equations (11) is known, then another set can be obtained by changing the signs of $\tau$, $q_1$, and $r_1$ in the former set. On making these changes in (41) we thus obtain the asymptotic solutions of (11) which approach zero as $\tau$ approaches $-\infty$, viz.,

$$\left.\begin{aligned} p_1 &= +\sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{k\rho_1\tau}\, u_{1k}^{(j)}(-\tau)\,\varepsilon^{j-1}\gamma^j, \\ q_1 &= -\sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{k\rho_1\tau}\, v_{1k}^{(j)}(-\tau)\,\varepsilon^{j-1}\gamma^j, \\ r_1 &= -\sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{k\rho_1\tau} w_{1k}^{(j)}(-\tau)\,\varepsilon^{j-1}\gamma^j. \end{aligned}\right\} \tag{42}$$

Upon substituting (41) and (42) in (10), and returning to the original variables $\xi, \eta, \zeta$ through the substitutions (4) and (2), the asymptotic solutions of the original differential equations (1) in terms of $\tau$ are found to be

$$\left.\begin{aligned} \xi &= \xi_0 + \bar{x}_1 + \sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{\mp k\rho_1\tau}\, u_{1k}^{(j)}(\pm\tau)\,\varepsilon^j\gamma^j, \\ \eta &= 0 + \bar{y}_1 \pm \sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{\mp k\rho_1\tau}\, v_{1k}^{(j)}(\pm\tau)\,\varepsilon^j\gamma^j, \\ \zeta &= 0 + \bar{z}_1 \pm \sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{\mp k\rho_1\tau} w_{1k}^{(j)}(\pm\tau)\,\varepsilon^j\gamma^j. \end{aligned}\right\} \tag{43}$$

Where the double signs occur, the upper signs give the solutions which approach the periodic orbits as $\tau$ approaches $+\infty$, and the lower signs give the solutions which approach the same periodic orbits as $\tau$ approaches $-\infty$.

*Asymptotic Orbits of Class B.*

We consider now the orbits which are asymptotic to the periodic orbits of Class B. The method of constructing these orbits is entirely similar to the one used in constructing the asymptotic orbits of Class A.

On putting $j=2$ and considering the differential equations obtained by equating the coefficients of the various powers of $\gamma$ in (11′), that is, in (11) when (33) and (34) have been substituted and $\gamma$ divided out, we obtain solutions which will give the asymptotic orbits, provided the arbitrary constants of integration are so chosen that conditions $(C_1)$ and $(C_2)$ are satisfied.

The differential equations arising from the terms independent of $\gamma$ in (11′) are the same as (12B), except for the superscript 0 on $p_2$, $q_2$, and $r_2$. Their solutions are therefore the same as (31) and (32). When the conditions $(C_1)$ and $(C_2)$ are imposed, we obtain

$$p_2^{(0)}=\alpha_2 e^{-\rho_2\tau}u_{22}, \quad q_2^{(0)}=-\alpha_2 e^{-\rho_2\tau}v_{22}, \quad r_2^{(0)}=0.$$

From the coefficients of $\gamma$ to the first degree in (11′) we obtain the differential equations which define $p_2^{(1)}$, $q_2^{(1)}$, and $r_2^{(1)}$. These equations have the same left members as (12B), except the superscript 1. Let the respective right members be denoted by $P_2^{(1)}$, $Q_2^{(1)}$, and $R_2^{(1)}$. Then

$$P_2^{(1)}=\varepsilon\alpha_2^2 e^{-2\rho_2\tau}U_{22}^{(1)}, \quad Q_2^{(1)}=\varepsilon\alpha_2^2 e^{-2\rho_2\tau}V_{22}^{(1)}, \quad R_2^{(1)}=0,$$

where $U_{22}^{(1)}$ and $V_{22}^{(1)}$ are functions of the same form as $u_{22}$ and $v_{22}$ respectively. The complementary functions of the differential equations are the same as (31) and (32), but let the arbitrary constants be denoted by $l_1^{(1)}, \ldots, l_6^{(1)}$. By using the method of the variation of parameters to obtain the complete solutions we have

$$\left.\begin{aligned}
&\dot{l}_1^{(1)}e^{\rho_2\tau}u_{21}+\dot{l}_2^{(1)}e^{-\rho_2\tau}u_{22}+\dot{l}_3^{(1)}u_{23}+\dot{l}_4^{(1)}(u_{24}+L\tau u_{23})=0,\\
&\dot{l}_1^{(1)}e^{\rho_2\tau}(\rho_2 u_{21}+\dot{u}_{21})+\dot{l}_2^{(1)}e^{-\rho_2\tau}(-\rho_2 u_{22}+\dot{u}_{22})+\dot{l}_3^{(1)}\dot{u}_{23}\\
&\qquad\qquad +\dot{l}_4^{(1)}[\dot{u}_{24}+L(u_{23}+\tau\dot{u}_{23})]=P_2^{(1)},\\
&\dot{l}_1^{(1)}e^{\rho_2\tau}v_{21}-\dot{l}_2^{(1)}e^{-\rho_2\tau}v_{22}+\dot{l}_3^{(1)}v_{23}+\dot{l}_4^{(1)}(v_{24}+L\tau v_{23})=0,\\
&\dot{l}_1^{(1)}e^{\rho_2\tau}(\rho_2 v_{21}+\dot{v}_{21})+\dot{l}_2^{(1)}e^{-\rho_2\tau}(\rho_2 v_{22}-\dot{v}_{22})+\dot{l}_3^{(1)}\dot{v}_{23}\\
&\qquad\qquad +\dot{l}_4^{(1)}[\dot{v}_{24}+L(v_{23}+\tau\dot{v}_{23})]=Q_2^{(1)},\\
&\dot{l}_5^{(1)}e^{i\omega\tau}w_{21}+\dot{l}_6^{(1)}e^{-i\omega\tau}w_{22}=0,\\
&\dot{l}_5^{(1)}e^{i\omega\tau}(i\omega w_{21}+\dot{w}_{21})-\dot{l}_6^{(1)}e^{-i\omega\tau}(i\omega w_{22}-\dot{w}_{22})=0.
\end{aligned}\right\} \quad (44)$$

The last two equations are independent of the first four. Since the determinant of the coefficients of $\dot{l}_5^{(1)}$ and $\dot{l}_6^{(1)}$ in these last two equations is different from zero, viz., $\Delta_2$, the only solutions for $\dot{l}_5^{(1)}$ and $\dot{l}_6^{(1)}$ are $\dot{l}_5^{(1)}=\dot{l}_6^{(1)}=0$.

The determinant of the coefficients of $\dot{l}_1^{(1)}, \ldots, \dot{l}_4^{(1)}$ is different from zero, viz., $\Delta_2$ equation (30), and therefore the first four equations of (44) can be solved for $\dot{l}_1^{(1)}, \ldots, \dot{l}_4^{(1)}$. The resulting solutions are

$$\dot{l}_k^{(1)} = \frac{\Delta_{2k}^{(1)}}{\Delta_2}, \qquad (k=1, \ldots, 4), \tag{45}$$

where $\Delta_{2k}^{(1)}$ is the determinant obtained by replacing the elements of the $k$-th row of $\Delta_2$ with 0, $P_2^{(1)}$, 0, and $Q_2^{(1)}$ respectively. Since the right members $P_2^{(1)}$ and $Q_2^{(1)}$ do not contain terms in $e^{\pm\rho_2\tau}$, the integrations of (45) for $l_1^{(1)}$ and $l_2^{(1)}$ will not contain terms in $\tau$ explicitly. Such terms arise, however, in the integration for $l_3^{(1)}$ and $l_4^{(1)}$, but when they are substituted in the complementary functions they cancel off. The complete solutions of the differential equations at this step are thus found to be

$$\left.\begin{aligned} p_2^{(1)} &= L_1^{(1)}e^{\rho_2\tau}u_{21} + L_2^{(1)}e^{-\rho_2\tau}u_{22} + L_3^{(1)}u_{23} + L_4^{(1)}(u_{24} + L\tau u_{23}) + \varepsilon\alpha_2^2 e^{-2\rho_2\tau}u_{22}^{(1)}(\tau), \\ q_2^{(1)} &= L_1^{(1)}e^{\rho_2\tau}v_{21} - L_2^{(1)}e^{-\rho_2\tau}v_{22} + L_3^{(1)}v_{23} + L_4^{(1)}(v_{24} + L\tau v_{23}) + \varepsilon\alpha_2^2 e^{-2\rho_2\tau}v_{22}^{(1)}(\tau), \\ r_2^{(1)} &= L_5^{(1)}e^{i\omega\tau}w_{21} + L_6^{(1)}e^{-i\omega\tau}w_{22}, \end{aligned}\right\} \tag{46}$$

where $L_1^{(1)} \ldots, L_6^{(1)}$ are the constants of integration, and $u_{22}^{(1)}(\tau)$ and $v_{22}^{(1)}(\tau)$ are functions similar to $u_{21}$ and $v_{21}$ respectively. From condition $(C_1)$ we have

$$L_1^{(1)} = L_3^{(1)} = L_4^{(1)} = L_5^{(1)} = L_6^{(1)} = 0,$$

and from condition $(C_2)$

$$L_2^{(1)} = -\varepsilon\alpha_2^2 u_{22}^{(1)}(0).$$

Then the solutions (46) become

$$\begin{aligned} p_2^{(1)} &= \varepsilon\alpha_2^2[e^{-\rho_2\tau}u_{21}^{(1)}(\tau) + e^{-2\rho_2\tau}u_{22}^{(1)}(\tau)], \\ q_2^{(1)} &= \varepsilon\alpha_2^2[e^{-\rho_2\tau}v_{21}^{(1)}(\tau) + e^{-2\rho_2\tau}v_{22}^{(1)}(\tau)], \\ r_2^{(1)} &= 0, \end{aligned}$$

where $u_{2k}^{(1)}(\tau)$ and $v_{2k}^{(1)}(\tau)$, $(k=1, 2)$, are similar to $u_{21}$ and $v_{21}$ respectively.

The remaining steps of the integration can be carried on in the same way. Proceeding by induction to the general term, we find that

$$p_2^{(\nu)} = \varepsilon^\nu\alpha_2^{\nu+1}\sum_{k=1}^{\nu+1} e^{-k\rho_2\tau}u_{2k}^{(\nu)}(\tau), \quad q_2^{(\nu)} = \varepsilon^\nu\alpha_2^{\nu+1}\sum_{k=1}^{\nu+1} e^{-k\rho_2\tau}v_{2k}^{(\nu)}(\tau), \quad r_2^{(\nu)} = 0,$$

where the functions $u_{2k}^{(\nu)}(\tau)$ and $v_{2k}^{(\nu)}(\tau)$ have the same form as $u_{21}$ and $v_{21}$ respectively. When these terms are substituted in (34) and (33), we find the solutions of (11) to be

$$p_2 = \sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{-k\rho_2\tau}u_{2k}^{(j)}(\tau)\varepsilon^{j-1}\alpha_2^j\gamma^j, \quad q_2 = \sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{-k\rho_2\tau}v_{2k}^{(j)}(\tau)\varepsilon^{j-1}\alpha_2^j\gamma^j, \quad r_2 = 0. \tag{47}$$

As in the solutions (41) we may put $\alpha_2 = 1$. These equations (47) are the

asymptotic solutions which approach zero as $\tau$ approaches $+\infty$. By changing the signs of $\tau$ and $q_2$ as in equations (42), we obtain

$$p_2=+\sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{k\rho_2\tau}u_{2k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j, \quad q_2=-\sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{k\rho_2\tau}v_{2k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j, \quad r_2=0, \quad (48)$$

which are the asymptotic solutions approaching zero as $\tau$ approaches $-\infty$. Since $r_2=0$ in (47) and (48), the asymptotic orbits are of two dimensions and are complanar with the periodic orbits of Class B.

In terms of the original variables $\xi$, $\eta$, $\zeta$, these asymptotic solutions of equations (1) are

$$\left.\begin{aligned}\xi&=\xi_0+\bar{x}_2+\sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{\mp k\rho_2\tau}u_{2k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j,\\ \eta&=0+\bar{y}_2\pm\sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{\mp k\rho_2\tau}v_{2k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j,\\ \zeta&=0.\end{aligned}\right\} \quad (49)$$

These solutions approach the periodic orbits of Class B as $\tau$ approaches $+\infty$ or $-\infty$ according respectively as the upper or lower signs are taken.

## § 9. Geometrical Considerations.

The asymptotic solutions (43) and (49) contain the three undetermined constants $t_0$, $\varepsilon$, and $\gamma$. The constant $t_0$ denotes the initial time and may be put equal to zero without loss of generality. The parameters $\varepsilon$ and $\gamma$ are the respective scale factors of the periodic and asymptotic orbits. The physical interpretation of the parameter $\varepsilon$ has already been referred to in the latter part of § 4. From the way in which the initial conditions were chosen in $(C_2)$, it is evident that $\varepsilon\gamma$ denotes the $\xi$-component of the infinitesimal body's initial displacement from the periodic orbit.

Let us now consider the directions in which the asymptotic orbits approach the periodic orbits. We shall discuss only the orbits which approach the periodic orbits of Class A as $\tau$ approaches $+\infty$, since the discussion is entirely similar for the other asymptotic orbits.

As $\tau$ becomes very large, the most important terms of the solutions of (11) are those in $e^{-\rho_1\tau}$. These terms arise only from the complementary functions of the differential equations which define the various $p_1^{(k)}$, $q_1^{(k)}$, and $r_1^{(k)}$ in (34). Neglecting the explicit values of the constants of integration which are associated with $e^{-\rho_1\tau}$ at each stage of the integration, we find that the predominating terms of the solutions as $\tau$ approaches $+\infty$ are

$$\begin{aligned}p_1&=+e^{-\rho_1\tau}u_{14}[K_4^{(0)}\gamma+K_4^{(1)}\gamma^2+K_4^{(2)}\gamma^3+\dots.],\\ q_1&=-e^{-\rho_1\tau}v_{14}[K_4^{(0)}\gamma+K_4^{(1)}\gamma^2+K_4^{(2)}\gamma^3+\dots.],\\ r_1&=-e^{-\rho_1\tau}w_{14}[K_4^{(0)}\gamma+K_4^{(1)}\gamma^2+K_4^{(2)}\gamma^2+\dots.],\end{aligned}$$

where $K_4^{(0)}$, $K_4^{(1)}$, ...., are the constants of integration.

The projection on the $\xi\eta$-plane of the direction of the approach is

$$\lim_{\tau=+\infty} \frac{dq_1}{d\tau} \Big/ \frac{dp_1}{d\tau} = \lim_{\tau=+\infty} \frac{\dot{v}_{14}-\rho_1 v_{14}}{\rho_1 u_{14}-\dot{u}_{14}}.$$

This limit is independent of $\gamma$, but, since $u_{14}$ and $v_{14}$ contain sines and cosines of multiples of $\sqrt{A}\tau$, it is indeterminate. The projections on the $\xi\zeta$- and $\eta\zeta$-planes are likewise found to be indeterminate. Similar results are obtained for the orbits which approach the periodic orbits of Class A as $\tau$ approaches $-\infty$, and also for the corresponding orbits of Class B.

## § 10. Illustrative Examples.

We shall conclude this article with numerical examples and diagrams of the periodic and asymptotic orbits which have been discussed in the preceding sections. In these examples the ratio of the finite masses is ten to one, or $1-\mu=10/11$ and $\mu=1/11$, being the ratio used in the particular periodic orbits already mentioned in § 4.

The solutions for the asymptotic orbits have been carried out to the third degree in $\varepsilon$ for orbits of Class A and to the second degree in $\varepsilon$ for orbits of Class B, but only to the first degree in $\gamma$ for both classes of orbits.

In the numerical results that are to be found in the tables which follow, $\varepsilon$ has the value 0.5 for Class A and 0.01 for Class B while $\gamma=0.1$ for both classes. The values of $\varepsilon x_j$, $\varepsilon y_j$, and $\varepsilon z_j$ $(j=1, 2)$ in the tables are the coordinates for the periodic orbits for the various values of $\tau$ indicated. The values for $\varepsilon\gamma p_j$, $\varepsilon\gamma q_j$, and $\varepsilon\gamma r_j$ are the amounts which must be added to the coordinates for the periodic orbits in order to obtain the asymptotic orbits. For $j=2$, $z_j=r_j=0$.

The periodic orbits are represented by the heavy lines in the diagrams, and the asymptotic orbits by the dotted lines. The arrows indicate the direction of motion. No arrows appear in the periodic orbits in Figs. 2, 5, and 8, as in these projections the infinitesimal body oscillates up and down along the same curve. The origin of coordinates is taken at the equilibrium point marked in the diagram, and the axes are parallel to the rotating $\xi\eta\zeta$ axes (see equations (2)). The unit of measurement is indicated in each diagram.

The following results have been obtained.

### *Orbits of Class A.*

*Equilibrium point (a).*

$$\varepsilon p_1 = e^{-\rho_1\tau}\gamma[\varepsilon+\varepsilon^3(-0.29+0.29\cos 2\sqrt{A}\tau-1.08\sin 2\sqrt{A}\tau)+\dots.],$$
$$\varepsilon q_1 = e^{-\rho_1\tau}\gamma[-0.75\,\varepsilon+\varepsilon^3(0.22-0.61\cos 2\sqrt{A}\tau-0.56\sin 2\sqrt{A}\tau)+\dots.],$$
$$\varepsilon r_1 = e^{-\rho_1\tau}\gamma[\varepsilon^2(1.58\cos\sqrt{A}\tau+0.91\sin\sqrt{A}\tau)+\dots.],$$
$$\rho_1 = 1.83-0.23\,\varepsilon^2+\dots., \quad A=2.548.$$

TABLE 3.

$\varepsilon=0.5,\ \gamma=0.1.$

| $\tau$ | $\epsilon x_1$ | $\epsilon y_1$ | $\epsilon z_1$ | $\epsilon\gamma p_1$ | $\epsilon\gamma q_1$ | $\epsilon\gamma r_1$ |
|---|---|---|---|---|---|---|
| 0 | —.041 | 0 | 0 | .0500 | —.0420 | .0400 |
| .1 | —.043 | —.008 | .043 | .0385 | —.0369 | .0359 |
| .2 | —.047 | —.015 | .085 | .0295 | —.0309 | .0315 |
| .3 | —.054 | —.021 | .125 | .0223 | —.0259 | .0270 |
| .4 | —.064 | —.026 | .163 | .0173 | —.0212 | .0227 |
| .5 | —.074 | —.028 | .198 | .0136 | —.0173 | .0187 |
| .6 | —.084 | —.028 | .228 | .0111 | —.0142 | .0155 |
| .7 | —.095 | —.025 | .254 | .0093 | —.0107 | .0118 |
| .8 | —.104 | —.021 | .274 | .0080 | —.0083 | .0091 |
| .9 | —.111 | —.015 | .289 | .0071 | —.0064 | .0067 |
| 1.0 | —.116 | —.007 | .297 | .0065 | —.0048 | .0048 |
| 1.2 | —.115 | .009 | .296 | .0056 | —.0029 | .0020 |
| 1.4 | —.103 | .022 | .271 | .0045 | —.0020 | .0003 |
| 1.6 | —.083 | .028 | .223 | .0034 | —.0015 | —.0005 6 |
| 1.8 | —.062 | .028 | .157 | .0025 | —.0013 | —.0009 1 |
| 2.0 | —.046 | .014 | .078 | .0017 | —.0011 | —.0009 5 |
| 2.4 | —.048 | —.016 | —.092 | .0005 9 | —.0006 | —.0006 5 |
| 2.8 | —.087 | —.027 | —.233 | .0002 3 | —.0002 7 | —.0003 0 |
| 3.2 | —.116 | —.006 | —.298 | .0001 4 | —.0000 9 | —.0000 9 |
| 3.6 | —.102 | .022 | —.269 | .0000 92 | —.0000 7 | —.0000 4 |
| 4.0 | —.060 | .024 | —.151 | .0000 48 | —.0000 3 | —.0000 18 |
| 4.4 | —.041 | —.003 | .014 | .0000 15 | —.0000 2 | —.0000 16 |

The projections of the above orbits on the coordinate planes are given in Figs. 1, 2, and 3.

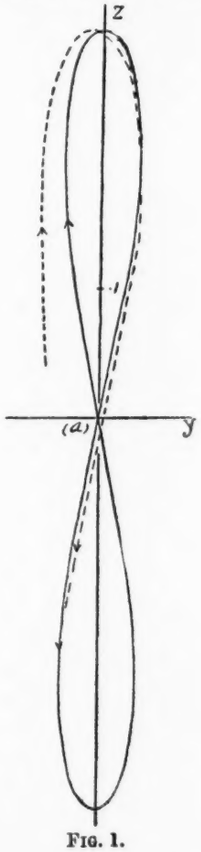


FIG. 1.

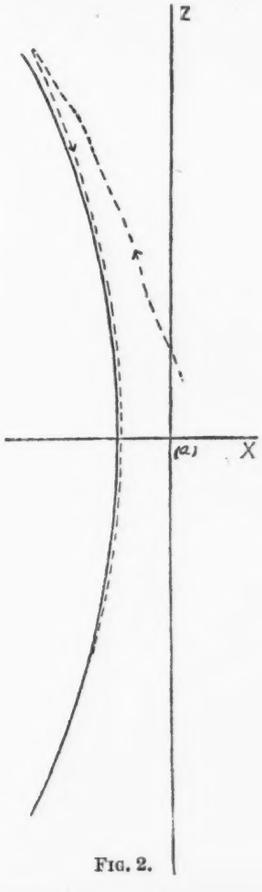


FIG. 2.

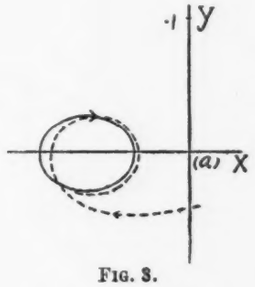


FIG. 3.

*Equilibrium point (b).*

$$\varepsilon p_1 = e^{-\rho_1 \tau}\gamma[\varepsilon+\varepsilon^3(0.222-0.222\cos 2\sqrt{A}\tau-0.454\sin 2\sqrt{A}\tau)+\dots.],$$
$$\varepsilon q_1 = e^{-\rho_1 \tau}\gamma[0.40\,\varepsilon+\varepsilon^2(0.089-0.182\cos 2\sqrt{A}\tau-0.037\sin 2\sqrt{A}\tau)+\dots.],$$
$$\varepsilon r_1 = e^{-\rho_1 \tau}\gamma[\varepsilon(-0.345\sin\sqrt{A}\tau-0.523\cos\sqrt{A}\tau)+\dots.],$$
$$\rho_1 = 3.366-2.837\,\varepsilon^2+\dots., \quad A=6.510.$$

TABLE 4.

$\varepsilon=0.5, \quad \gamma=0.1.$

| $\tau$ | $\varepsilon x_1$ | $\varepsilon y_1$ | $\varepsilon z_1$ | $\varepsilon\gamma p_1$ | $\varepsilon\gamma q_1$ | $\varepsilon\gamma r_1$ |
|---|---|---|---|---|---|---|
| 0 | .0135 | 0 | 0 | .0500 | .0188 | −.0131 |
| .1 | .0147 | .0022 | .0474 | .0300 | .0119 | −.0094 0 |
| .2 | .0178 | .0038 | .0917 | .0186 | .0073 8 | −.0062 6 |
| .3 | .0221 | .0045 | .1355 | .0118 | .0051 7 | −.0038 6 |
| .4 | .0266 | .0040 | .1602 | .0077 9 | .0034 5 | −.0022 6 |
| .5 | .0300 | .0025 | .1799 | .0051 9 | .0022 7 | −.0009 57 |
| .6 | .0315 | .0004 | .1878 | .0034 9 | .0014 7 | −.0005 76 |
| .7 | .0307 | −.0019 | .1837 | .0023 1 | .0009 36 | −.0002 24 |
| .8 | .0278 | −.0036 | .1677 | .0014 9 | .0005 74 | −.0000 454 |
| .9 | .0236 | −.0045 | .1594 | .0009 34 | .0003 47 | +.0000 350 |
| 1.0 | .0206 | −.0044 | .1047 | .0005 83 | .0002 13 | +.0000 616 |
| 1.2 | .0136 | −.0007 | .0150 | .0002 04 | .0000 762 | +.0000 497 |
| 1.4 | .0166 | .0034 | −.0784 | .0000 750 | .0000 309 | +.0000 363 |
| 1.6 | .0252 | .0043 | −.1519 | .0000 308 | .0000 136 | +.0000 0936 |
| 1.8 | .0312 | .0011 | −.1867 | .0000 137 | .0000 0590 | +.0000 0257 |
| 2.0 | .0289 | −.0032 | −.1837 | .0000 0594 | .0000 0232 | +.0000 0056 6 |
| 2.2 | .0205 | −.0044 | −.1169 | .0000 0233 | .0000 0081 0 | −.0000 0021 6 |
| 2.4 | .0140 | −.0014 | −.0301 | .0000 0086 6 | .0000 0030 8 | −.0000 0002 2 |
| 2.6 | .0156 | .0029 | .0645 | .0000 0029 9 | .0000 0012 4 | −.0000 0001 9 |

The projections of the above orbits on the coordinate planes are given in Figs. 4, 5, and 6.

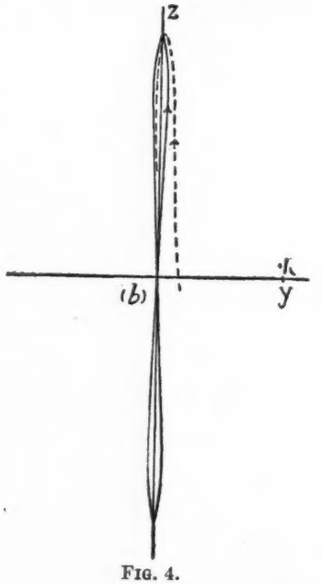


FIG. 4.

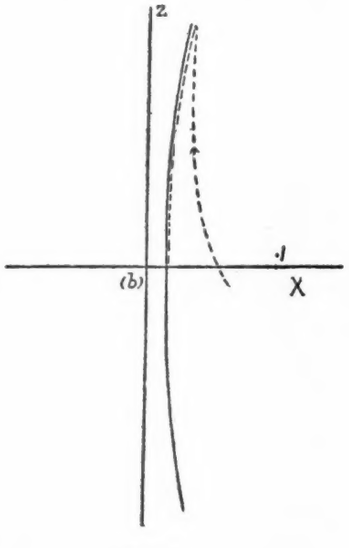


FIG. 5.

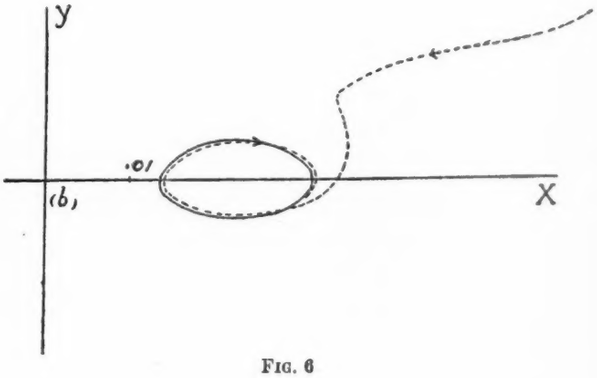


FIG. 6

*Equilibrium point (c).*

$$\varepsilon p_1 = e^{-\rho_1 \tau}\gamma[\varepsilon+\varepsilon^3(-0.104+0.104\cos 2\sqrt{A}\tau-0.817\sin 2\sqrt{A}\tau)+\dots.],$$
$$\varepsilon q_1 = e^{-\rho_1 \tau}\gamma[3.08\,\varepsilon+\varepsilon^3(-0.320-0.787\cos 2\sqrt{A}\tau-0.106\sin 2\sqrt{A}\tau)+\dots.],$$
$$\varepsilon r_1 = e^{-\rho_1 \tau}\gamma[\varepsilon(-0.719\sin\sqrt{A}\tau-3.145\cos\sqrt{A}\tau)+\dots.],$$
$$\rho_1 = 0.457-0.025\,\varepsilon^2+\dots., \quad A=1.082.$$

TABLE 5.

$\varepsilon=0.5$, $\gamma=0.1$.

| $\tau$ | $\varepsilon x_1$ | $\varepsilon y_1$ | $\varepsilon z_1$ | $\varepsilon\gamma p_1$ | $\varepsilon\gamma q_1$ | $\varepsilon\gamma r_1$ |
|---|---|---|---|---|---|---|
| 0 | .0050 | 0 | 0 | .0500 | .1402 | —.0786 |
| .2 | .0099 | .0228 | .0989 | .0418 | .1279 | —.0733 |
| .4 | .0238 | .0418 | .1939 | .0349 | .1180 | —.0657 |
| .6 | .0443 | .0536 | .2803 | .0297 | .1098 | —.0560 |
| .8 | .0678 | .0563 | .3547 | .0264 | .1030 | —.0455 |
| 1.0 | .0905 | .0493 | .4138 | .0245 | .0960 | —.0346 |
| 1.2 | .1084 | .0340 | .4550 | .0236 | .0893 | —.0239 |
| 1.4 | .1185 | .0129 | .4766 | .0234 | .0825 | —.0139 |
| 1.6 | .1190 | —.0105 | .4781 | .0233 | .0754 | —.0051 |
| 1.8 | .1099 | —.0320 | .4584 | .0229 | .0680 | .0026 |
| 2.0 | .0927 | —.0481 | .4190 | .0221 | .0612 | .0088 |
| 2.2 | .0703 | —.0560 | .3619 | .0209 | .0543 | .0135 |
| 2.4 | .0466 | —.0543 | .2890 | .0191 | .0481 | .0168 |
| 2.6 | .0256 | —.0434 | .2035 | .0169 | .0426 | .0187 |
| 2.8 | .0110 | —.0251 | .1094 | .0147 | .0377 | .0194 |
| 3.0 | .0051 | —.0025 | .0105 | .0123 | .0341 | .0191 |
| 3.4 | .0220 | .0400 | —.1843 | .0086 | .0289 | .0161 |
| 3.8 | .0653 | .0564 | —.3475 | .0064 5 | .0251 | .0113 |
| 4.2 | .1069 | .0359 | —.4517 | .0057 3 | .0218 | .0060 |
| 4.6 | .1194 | —.0080 | —.4790 | .0056 2 | .0184 | .0014 |
| 5 | .1101 | —.0318 | —.4723 | .0051 0 | .0151 | .0015 |
| 5.4 | .0491 | —.0549 | —.2971 | .0046 6 | .0119 | —.0040 |
| 5.8 | .0121 | —.0272 | —.1195 | .0035 9 | .0093 | —.0047 |
| 6.2 | .0081 | .0182 | .0787 | .0014 5 | .0076 | —.0044 |

The projections of the above orbits on the coordinate planes are given in Figs. 7, 8, and 9.

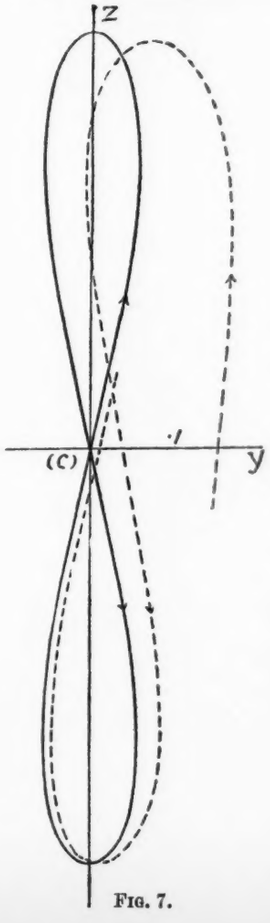


FIG. 7.

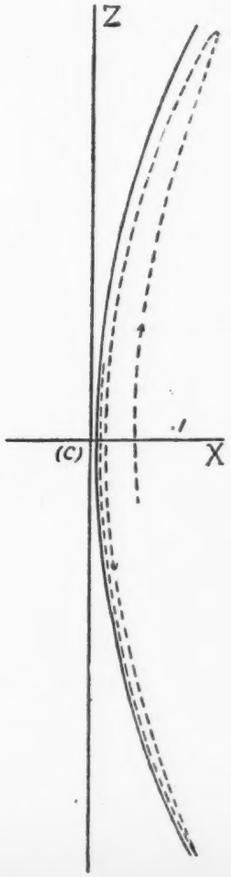


FIG. 8.

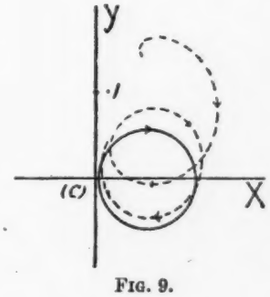


FIG. 9.

*Equilibrium point* $(a)$.

$$\varepsilon p_2 = e^{-\rho_2 \tau}\gamma[\varepsilon+\varepsilon^2(-1.20+1.20\cos\sigma\tau+3.95\sin\sigma\tau)+\dots.],$$
$$\varepsilon q_2 = e^{-\rho_2 \tau}\gamma[0.75\,\varepsilon-\varepsilon^2(0.90+4.65\cos\sigma\tau+2.46\sin\sigma\tau)+\dots.],$$
$$\rho_2 = 1.833+()\varepsilon^2+\dots., \quad \sigma=1.68.$$

TABLE 6.

$\varepsilon=0.01, \quad \gamma=0.1.$

| $\tau$ | $\varepsilon x_2$ | $\varepsilon y_2$ | $\varepsilon\gamma p_2$ | $\varepsilon\gamma q_2$ |
|---|---|---|---|---|
| 0 | .0100 | 0 | .0010 00 | .0007 13 |
| .2 | .0094 | —.0088 | .0007 00 | .0004 77 |
| .4 | .0078 | —.0166 | .0004 90 | .0003 31 |
| .6 | .0053 | —.0225 | .0003 42 | .0002 31 |
| .8 | .0023 | —.0259 | .0002 38 | .0001 63 |
| 1.0 | —.0011 | —.0264 | .0001 63 | .0001 16 |
| 1.2 | —.0043 | —.0240 | .0001 13 | .0000 820 |
| 1.4 | —.0070 | —.0189 | .0000 776 | .0000 581 |
| 1.6 | —.0090 | —.0117 | .0000 534 | .0000 412 |
| 1.8 | —.0099 | —.0031 | .0000 361 | .0000 289 |
| 2.0 | —.0098 | .0058 | .0000 248 | .0000 203 |
| 2.2 | —.0085 | .0140 | .0000 169 | .0000 140 |
| 2.4 | —.0063 | .0207 | .0000 117 | .0000 0970 |
| 2.6 | —.0034 | .0250 | .0000 0809 | .0000 0664 |
| 2.8 | —.0001 | .0266 | .0000 0560 | .0000 0452 |
| 3.0 | .0032 | .0252 | .0000 0392 | .0000 0306 |
| 3.2 | .0062 | .0210 | .0000 0275 | .0000 0208 |
| 3.4 | .0084 | .0144 | .0000 0193 | .0000 0141 |
| 3.6 | .0097 | .0062 | .0000 0136 | .0000 0095 |
| 3.8 | .0100 | .0027 | .0000 0094 | .0000 0065 |

These orbits are shown in Fig. 10.

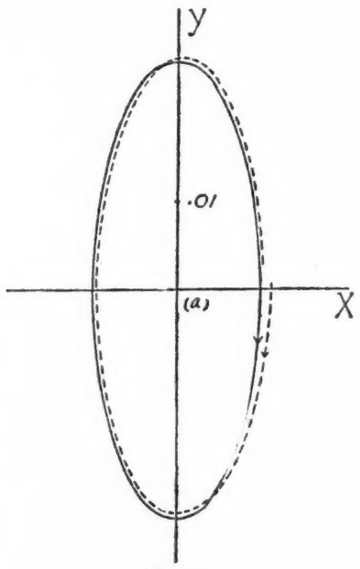


FIG. 10.

*Equilibrium point (b).*

$$\varepsilon p_2 = e^{-\rho_2 \tau}\gamma[\varepsilon+\varepsilon^2(0.69-0.69\cos\sigma\tau-2.47\sin\sigma\tau)+\dots.],$$
$$\varepsilon q_2 = e^{-\rho_2 \tau}\gamma[0.40\,\varepsilon+\varepsilon^2(0.28+4.60\cos\sigma\tau+3.07\sin\sigma\tau)+\dots.],$$
$$\rho_2 = 3.366+()\varepsilon^2+\dots., \quad \sigma=2.61.$$

TABLE 7.

$\varepsilon=0.01$, $\gamma=0.1$.

| $\tau$ | $\varepsilon x_2$ | $\varepsilon y_2$ | $\varepsilon\gamma p_2$ | $\varepsilon\gamma q_2$ |
|---|---|---|---|---|
| 0 | .0100 | 0 | .0010 00 | .0004 49 |
| .1 | .0097 | −.0103 | .0007 10 | .0003 70 |
| .2 | .0087 | −.0199 | .0005 04 | .0002 34 |
| .3 | .0071 | −.0282 | .0003 59 | .0001 66 |
| .4 | .0050 | −.0345 | .0002 55 | .0001 17 |
| .5 | .0026 | −.0385 | .0001 82 | .0000 83 |
| .6 | .0000 4 | −.0399 | .0001 31 | .0000 58 |
| .7 | −.0025 | −.0386 | .0000 933 | .0000 398 |
| .8 | −.0050 | −.0347 | .0000 670 | .0000 275 |
| .9 | −.0070 | −.0284 | .0000 481 | .0000 190 |
| 1.0 | −.0086 | −.0202 | .0000 345 | .0000 131 |
| 1.2 | −.0100 | −.0003 | .0000 178 | .0000 063 |
| 1.4 | −.0087 | .0196 | .0000 092 | .0000 031 |
| 1.6 | −.0051 | .0344 | .0000 047 | .0000 016 |
| 1.8 | −.0001 | .0399 | .0000 024 | .0000 0087 |
| 2.0 | +.0049 | .0348 | .0000 012 | .0000 0047 |
| 2.2 | +.0086 | .0204 | .0000 006 | .0000 0026 |
| 2.4 | .0100 | .0006 | .0000 003 | .0000 0014 |

These orbits are shown in Fig. 11.

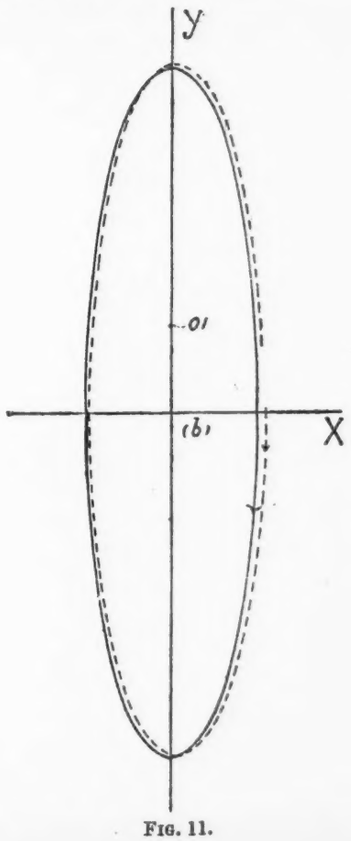


FIG. 11.

*Equilibrium point* ($c$).

$$\varepsilon p_2 = e^{-\rho_2 \tau}\gamma[\varepsilon+\varepsilon^2(25-25\cos\sigma\tau-23\sin\sigma\tau)+\dots.],$$
$$\varepsilon q_2 = e^{-\rho_2 \tau}\gamma[3.08\,\varepsilon+\varepsilon^2(77-56\cos\sigma\tau+31\sin\sigma\tau)+\dots.],$$
$$\rho_2 = 0.475+()\varepsilon^2+\dots., \quad \sigma=1.07.$$

TABLE 8.

$\varepsilon=0.01, \quad \gamma=0.1.$

| $\tau$ | $\varepsilon x_2$ | $\varepsilon y_2$ | $\varepsilon\gamma p_2$ | $\varepsilon\gamma q_2$ |
|---|---|---|---|---|
| 0 | .0100 | 0 | .0010 0 | .0032 9 |
| .1 | .0099 | —.0022 | .0009 3 | .0031 7 |
| .2 | .0098 | —.0043 | .0008 7 | .0030 6 |
| .3 | .0095 | —.0063 | .0008 1 | .0029 7 |
| .4 | .0091 | —.0083 | .0007 6 | .0028 7 |
| .5 | .0086 | —.0103 | .0007 2 | .0027 9 |
| .6 | .0080 | —.0120 | .0006 8 | .0027 0 |
| .7 | .0073 | —.0137 | .0006 5 | .0026 2 |
| .8 | .0066 | —.0152 | .0006 3 | .0025 4 |
| .9 | .0057 | —.0165 | .0006 0 | .0024 6 |
| 1.0 | .0048 | —.0176 | .0005 8 | .0023 9 |
| 1.5 | —.0003 | —.0199 | .0005 4 | .0020 5 |
| 2.0 | —.0054 | —.0169 | .0004 6 | .0017 1 |
| 2.5 | —.0089 | —.0090 | .0004 2 | .0013 7 |
| 3.0 | —.0100 | .0014 | .0003 7 | .0010 6 |
| 3.5 | —.0082 | .0140 | .0003 0 | .0007 8 |
| 4.0 | —.0042 | .0183 | .0002 3 | .0005 7 |
| 4.5 | .0010 | .0200 | .0001 4 | .0004 1 |
| 5.0 | .0060 | .0162 | .0001 2 | .0003 0 |
| 5.5 | .0092 | .0078 | .0000 8 | .0002 3 |
| 6.0 | .0099 | —.0027 | .0000 6 | .0001 9 |

These orbits are shown in Fig. 12.

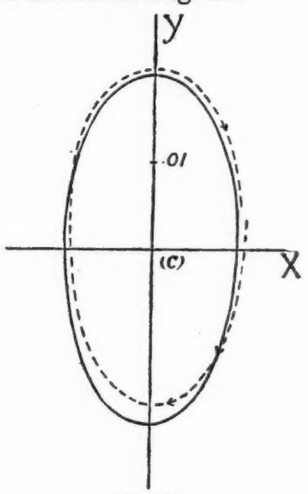


FIG. 12.

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# *Concerning the Invariant Theory of Involutions of Conics.*

By Wayne Sensenig.

## § 1. Introduction.

The complete system of invariants of a single conic

$$f=a_x^2=a_{200}x_1^2+2a_{110}x_1x_2+\ldots.$$

consists of four members: $f=a_x^2$, $L=a_u^2$, $D=a_a^2$, and $u_x$. The complete system of invariants of two conics consists of twenty forms and this was derived first by Gordan and was first published in Clebsch's "Vorlesungen über Geometrie."* H. F. Baker derived the complete system for three, and incidentally two, conics.† Ciamberlini has published a determination of a complete system for three conics.‡

The object of this paper is to derive and reduce in terms of the system of two conics $a_x^2$, $b_x^2$ the complete simultaneous system of the involution $K=a_x^2+kb_x^2$ and the harmonic conic $F=(\alpha\beta x)^2$. The complete system of $K$ and $F$ is expressed in terms of the complete system of the two base conics $f=a_x^2$ and $g=b_x^2$, which determine the configuration.

In Bôchers "Introduction to Higher Algebra," p. 164, it is shown that if two conics $f$, $g$ intersect in four distinct points there exists a (non-singular) collineation which will reduce $f$ and $g$ simultaneously into the normal forms given below. Note that the equalities hold only by virtue of the equations of the collineation:

$$a_x^2=ax_1^2+bx_2^2+cx_3^2, \quad b_x^2=px_1^2+qx_2^2+rx_3^2. \tag{1}$$

In Table I, I give the well-known symbolical forms of the system of $f$ and $g$, and also the normal system which I have computed from these symbolical forms by particularizing the coefficients of $f$ and $g$, respectively, according to the following scheme:

$$a_{110}=a_{101}=a_{011}=0, \quad a_{200}=a, \quad a_{020}=b, \quad a_{002}=c,$$
$$b_{110}=b_{101}=b_{011}=0, \quad b_{200}=p, \quad b_{020}=q, \quad b_{002}=r.$$

Throughout the paper we have used the algorism for ternary transvection given by O. E. Glenn in the Transactions of the American Mathematical Society,

* Cf. Osgood, American Journal of Mathematics, Vol. XIV (1892), p. 262, "System of two Simultaneous Ternary Quadratic Forms."

† *Cambridge Philosophical Transactions*, Vol. XV (1889–93), p. 62.

‡ *Giornale di Matematiche*, Vol. XXIV (1885–86), p. 141.

Vol. XVII (1916).* This algorism is analytical but it will suffice here to describe the transvectant $\tau$ of index $\binom{ij}{kl}$ of the two forms

$$\phi = Aa_{1x}a_{2x}\ldots a_{rx}\alpha_{1u}\ldots\alpha_{su}, \quad \psi = Bb_{1x}\ldots b_{px}\beta_{1u}\ldots\beta_{qu},$$

where $A$ and $B$ are constants, as $AB/\mu$ times the sum of all terms gotten by forming in the product $\phi\psi$, in all possible $(\mu)$ ways, $i$ convolutions of the type $a_{1\beta_1}$, $j$ of type $(a_1b_1u)$, $k$ of the type $\alpha_{1b_1}$, and $l$ of type $(\alpha_1\beta_1x)$. Then $\tau$ is abbreviated as

$$\tau = (\phi, \psi)^{ij}_{kl}.$$

In deriving the system of $K$ and $F$ we have evaluated each member as a transvectant and then reduced each form to a rational integral function of the members of the fundamental system of $a_x^2$ and $b_x^2$, expressing each one in the form of a polynomial in $k$. These reductions were performed by means of the following fundamental ternary identities, and their modifications:

$$\left.\begin{aligned} (abc)d_x - (abd)c_x + (acd)b_x - (bcd)a_x &\equiv 0, \\ (abu)(\alpha\beta x) &= \begin{vmatrix} a_\alpha & b_\alpha & u_\alpha \\ a_\beta & b_\beta & u_\beta \\ a_x & b_x & u_x \end{vmatrix}. \end{aligned}\right\} \tag{2}$$

The result for each case was then checked by direct verification, for the system of normal forms given in Table I. Table II contains the summary of the results of the reductions for all twenty concomitants in the system for $K$ and $F$.

In Table III of the paper we give the fundamental system of the involution $a_x^2 + kb_x^2$ taken with a third general conic $c_x^2$. All concomitants in this system are expressed as rational integral functions of certain forms of the system for three conics as given by Baker; in fact in terms of the sixty-one forms of this system which are the simplest and therefore the most important in geometrical applications. This set of sixty-one concomitants is given below:

$f=a_x^2$, $g=b_x^2$, $\Phi_2=(acu)^2$, $\Phi_1=(bcu)^2$, $C_{12}=(acu)a_xc_x$, $C_{11}=(bcu)b_xc_x$, $C_1=a_xb_x(abu)$, $A_1=a_\gamma^2$, $A_2=b_\gamma^2$, $A_3=c_\alpha^2$, $A_4=c_\beta^2$, $C_{21}=a_\gamma a_x\gamma_u$, $C_{22}=b_\gamma b_x\gamma_u$, $C_{23}=c_\alpha c_x\alpha_u$, $C_{24}=c_\beta c_x\beta_u$, $C_{42}=(\alpha\gamma x)\alpha_u\gamma_u$, $C_{41}=(\beta\gamma x)\beta_u\gamma_u$, $F_2=(\alpha\gamma x)^2$, $F_1=(\beta\gamma x)^2$, $C_{51}=c_\alpha c_x\gamma_u(\alpha\gamma x)$, $C_{52}=c_\beta c_x\gamma_u(\beta\gamma x)$, $C_{53}=a_\gamma a_x\alpha_u(\alpha\gamma x)$, $L''=\gamma_u^2$, $C_{61}=a_\gamma c_x\gamma_u(acu)$, $C_{62}=b_\gamma c_x\gamma_u(bcu)$, $C_{63}=a_xc_\alpha\alpha_u(acu)$, $C_{64}=b_xc_\beta\beta_u(bcu)$, $G_1=a_\gamma c_\alpha a_xc_x(\alpha\gamma x)$, $G_2=b_\gamma c_\beta b_xc_x(\beta\gamma x)$, $\Gamma_2=a_\gamma c_\alpha u_\alpha u_\gamma(acu)$, $\Gamma_1=b_\gamma c_\beta u_\beta u_\gamma(bcu)$, $J=u_x$, $M=(abc)(abu)c_x$, $N=(abc)^2$, $O_1=a_\gamma b_x\gamma_u(abu)$, $O_2=a_xb_\gamma\gamma_u(abu)$, $O_3=b_xc_\alpha\alpha_u(bcu)$, $O_4=a_xc_\beta\beta_u(acu)$, $P=a_\gamma a_xb_\gamma b_x$, $Q_1=a_\gamma b_xc_x\gamma_u(abc)$, $Q_2=a_xb_\gamma c_x\gamma_u(abc)$,

* Cf. Glenn, "Theory of Invariants" (Boston, Ginn & Co., 1915), p. 227.

$R_1=(a'bc)(acu)(abu)a'_x$, $R_2=(ab'c)(bcu)(abu)b'_x$, $S_1=b_ac_a(bcu)$, $S_2=a_\beta c_\beta(acu)$, $S_3=a_\gamma b_\gamma(abu)$, $T_1=b_xb_\gamma\alpha_u(\alpha\gamma x)$, $T_2=b_xb_a\gamma_u(\alpha\gamma x)$, $T_3=a_\gamma a_x\beta_u(\beta\gamma x)$, $T_4=a_\beta a_x u_\gamma(\beta\gamma x)$, $T_5=b_\gamma b_x\beta_u(\beta\gamma x)$, $U_1=b_ab_\gamma(\alpha\gamma x)$, $U_2=a_\beta a_\gamma(\beta\gamma x)$, $V=a_xb_xc_x(abc)$, $W=a_\gamma b_\gamma c_x(abc)$, $X_1=b_\gamma c_a b_x c_x(\alpha\gamma x)$, $X_2=a_\gamma c_\beta a_x c_x(\beta\gamma x)$, $Y=(acu)(abu)(bcu)$, $Z_1=b_\gamma c_a u_a u_\gamma(bcu)$, $Z_2=a_\gamma c_\beta u_\beta u_\gamma(acu)$;

where those preceding $J$ are concomitants of two conics and those following $J$, of three conics.

## TABLE I.

### *Normal System of Two Conics.*

1. $f = a_x^2=a_x'^2=a_x''^2=\text{etc.}=ax_1^2+bx_2^2+cx_3^2.$
2. $g = b_x^2=b_x'^2=b_x''^2=\text{etc.}=px_1^2+qx_2^2+rx_3^2.$
3. $D = a_a^2=6abc.$
4. $D' = b_\beta^2=6pqr.$
5. $L = \alpha_u^2=2(bcu_1^2+acu_2^2+abu_3^2).$
6. $L' = \beta_u^2=2(qru_1^2+pru_2^2+pqu_3^2).$
7. $\Phi = (a_x^2, b_x^2)_{00}^{02}=(abu)^2=(br+cq)u_1^2+(ar+cp)u_2^2+(aq+bp)u_3^2.$
8. $C_1 = (a_x^2, b_x^2)_{00}^{01}=(abu)a_xb_x$
   $=(aq-bp)x_1x_2u_3+(cp-ar)x_1x_3u_2+(br-cq)x_2x_3u_1.$
9. $A_{122}=(a_x^2, \beta_u^2)_{00}^{20}=a_\beta^2=2(aqr+bpr+cpq).$
10. $C_2 = (a_x^2, \beta_u^2)_{00}^{10}=a_\beta a_x\beta_u=2(aqrx_1u_1+bprx_2u_2+cpqx_3u_3).$
11. $C_3 = (\alpha_u^2, b_x^2)_{10}^{00}=\alpha_b\alpha_ub_x=2(bcpx_1u_1+acqx_2u_2+abrx_3u_3).$
12. $A_{112}=(\alpha_u^2, b_x^2)_{20}^{00}=\alpha_b^2=2(bcp+acq+abr).$
13. $C_4 = (\alpha_u^2, \beta_u^2)_{01}^{00}=(\alpha\beta x)\alpha_u\beta_u$
    $=4[cr(bp-aq)u_1u_2x_3+bq(ar-cp)u_1u_3x_2+ap(cq-br)u_2u_3x_1].$
14. $F = (\alpha_u^2, \beta_u^2)_{02}^{00}=(\alpha\beta x)^2$
    $=4[ap(br+cq)x_1^2+bq(ar+cp)x_2^2+cr(bp+aq)x_3^2].$
15. $C_5 = (\alpha_u^2, b_x^2\beta_u^2)_{11}^{00}=\alpha_b(\alpha\beta x)b_x\beta_u$
    $=4[aqr(br-cq)x_2x_3u_1+bpr(cp-ar)x_1x_3u_2+cpq(aq-bp)x_1x_2u_3].$
16. $C_6 = (a_x^2, b_x^2\beta_u^2)_{00}^{11}=a_\beta(abu)b_x\beta_u$
    $=2[p^2(cq-br)u_2u_3x_1+q^2(ar-cp)u_1u_3x_2+r^2(bp-aq)u_1u_2x_3].$
17. $C_7 = (a_x^2a_u^2, b_x^2)_{10}^{01}=\alpha_b(abu)a_x\alpha_u$
    $=2[a^2(cq-br)u_2u_3x_1+b^2(ar-cp)u_1u_3x_2+c^2(bp-aq)u_1u_2x_3].$
18. $C_8 = (a_x^2\alpha_u^2, \beta_u^2)_{01}^{10}=a_\beta(\alpha\beta x)a_x\alpha_u$
    $=4[abr(aq-bp)x_1x_2u_3+acq(cp-ar)x_1x_3u_2+bcp(br-cq)x_2x_3u_1].$
19. $G = (fL, gL')_{11}^{10}=a_\beta\alpha_b(\alpha\beta x)a_xb_x$
    $=4[a^2qr(br-cq)+b^2pr(cp-ar)+c^2pq(aq-bp)]x_1x_2x_3.$
20. $\Gamma = (fL, gL')_{10}^{11}=a_\beta\alpha_b(abu)\alpha_u\beta_u$
    $=4[p^2bc(cq-br)+q^2ac(ar-cp)+r^2ab(bp-aq)]u_1u_2u_3.$

## § 2. REDUCTION OF THE SYSTEM OF $K$ AND $F$.

The complete system of $K$ and $F$ is obtained by forming and reducing the transvectants of $K$ and $F$ which correspond respectively to the transvectants of $f, g$ required to produce the system of $f, g$ (cf. Table I). Thus a member of the system of $f, g$ is $\Phi=(f,g)^{02}_{00}$. Hence the corresponding form of the required system is

$$\Phi'=(K,F)^{02}_{00}.$$

I have not included in this paper all of the reductions which were necessary to produce Table II; but only typical ones. The most complicated cases of all were $C'_5$, $G'$, and $\Gamma'$. We add that the reductions for 3, 4, 5, 6 in Table II had been given previously.*

### *The Form $\Phi'$.*

$$\begin{aligned}
\Phi'&=(K,F)^{02}_{00}=(a_x^2+kb_x^2,\,(\alpha\beta)_x^2)^{02}_{00}=(a_x^2,\,(\alpha\beta)_x^2)^{02}_{00}+k(b_x^2,\,(\alpha\beta)_x^2)^{02}_{00}\\
&=(a\widehat{\alpha\beta}u)^2+k(b\widehat{\alpha\beta}u)^2=(a_\alpha u_\beta-a_\beta u_\alpha)^2+k(b_\alpha u_\beta-b_\beta u_\alpha)^2\\
&=a_\alpha^2u_\beta^2+a_\beta^2u_\alpha^2-2a_\alpha a_\beta u_\alpha u_\beta+k(b_\alpha^2u_\beta^2+b_\beta^2u_\alpha^2-2b_\alpha b_\beta u_\alpha u_\beta)\\
&=DL'+A_{122}L-2(aa'a'')\,(a'a''u)\,a_\beta u_\beta\\
&\qquad+k[D'L+A_{112}L'-2(bb'b'')\,(b'b''u)\,b_\alpha u_\alpha]\\
&=DL'+A_{122}L-\tfrac{2}{3}DL'+k[D'L+A_{112}L'-\tfrac{2}{3}D'L]\\
&=\tfrac{1}{3}DL'+A_{122}L+k[\tfrac{1}{3}D'L+A_{112}L'].
\end{aligned}$$

### *The Form $C'_3$.*

$$\begin{aligned}
C'_3&=(a_u^2+2k(abu)^2+k^2\beta_u^2,\,(\gamma\delta)_x^2)^{00}_{10}, \text{ where } \alpha=\gamma \text{ and } \beta=\delta,\\
&=(a_u^2,\,(\gamma\delta)_x^2)^{00}_{10}+2k[\,(ab)_u^2,\,(\gamma\delta)_x^2]^{00}_{10}+k^2[\beta_u^2,\,(\gamma\delta)_x^2]^{00}_{10}\\
&=(\alpha\gamma\delta)a_u(\gamma\delta x)+2k(\widehat{ab\gamma\delta})\,(abu)\,(\gamma\delta x)+k^2(\beta\gamma\delta)\beta_u(\gamma\delta x).
\end{aligned}$$

1. $(\alpha\gamma\delta)a_u(\gamma\delta x)\equiv\frac{1}{2}(\alpha\gamma\delta)\{(\gamma\delta x)a_u-(a\delta x)\gamma_u\}=\frac{1}{2}(\alpha\gamma\delta)\,[(\alpha\gamma\delta)x_u-(\alpha\gamma x)\delta_u]$

$=\frac{1}{2}(\alpha\gamma\delta)^2x_u-\frac{1}{2}(\alpha\gamma\delta)\,(\alpha\gamma x)\,\delta_u.$

$(\alpha\gamma\delta)(\alpha\gamma x)\delta_u=(\alpha\widehat{a'''a^{IV}}\delta)(\alpha\widehat{a'''a^{IV}}x)\delta_u=(a'''_\alpha a^{IV}_\delta-a^{IV}_\alpha a'''_\delta)(a'''_\alpha a^{IV}_x-a^{IV}_\alpha a'''_x)\delta_u$

$=[(a'a''a''')(b'b''a^{IV})-(a'a''a^{IV})(b'b''a''')]\,[(a'a''a''')a^{IV}_x-(a'a''a^{IV})a'''_x]\delta_u$

$=[(a'a''a''')^2a^{IV}_\beta a^{IV}_x-(a'a''a''')\,(a'a''a^{IV})\,a^{IV}_\beta a'''_x$

$+(a'a''a^{IV})^2a'''_\beta a'''_x-(a'a''a''')\,(a'a''a^{IV})\,a'''_\beta a^{IV}_x]\delta_u$

$=D\cdot a^{IV}_\beta a^{IV}_x\beta_u-\frac{1}{3}D\cdot a^{IV}_\beta a^{IV}_x\beta_u+D\cdot a'''_\beta a'''_x\beta_u-\frac{1}{3}D\cdot a^{IV}_\beta a^{IV}_x\beta_u=\frac{4}{3}DC_2.$

$\therefore\ (\alpha\gamma\delta)\,(\gamma\delta x)a_u=\frac{2}{3}DA_{122}u_x-\frac{2}{3}DC_2$; since $(\alpha\gamma\delta)^2=\frac{4}{3}DA_{122}, (\beta\gamma\delta)^2=\frac{4}{3}D'A_{112}.$

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*"Algebra of Invariants," by Grace and Young, p. 293, *Cambridge University Press*, 1903.

2. $(\widehat{ab}\gamma\delta)(abu)(\gamma\delta x)=(a_\gamma b_\delta-a_\delta b_\gamma)(abu)(\alpha\beta x)=a_a b_\beta(abu)(\alpha\beta x)$
$-a_\beta b_\alpha(abu)(\alpha\beta x)=a_a b_\beta(abu)(\alpha\beta x)-k_3$
$=\frac{2}{3}DD'u_x-k_3$, since $a_a b_\beta(abu)(\widehat{ab'b''}x)=a_a b_\beta(abu)[b''_a b'_x-b'_a b''_x]$
$=a_a b''_a b_\beta b'_x(abu)-a_a b'_a b_\beta b''_x(abu)=(aa'a'')(a'a''b'')a_{(bu)}b_\beta b'_x$
$-(aa'a'')(a'a''b')a_{(bu)}b_\beta b''_x=\frac{1}{3}D(b''bu)(bb'b'')b'_x$
$-\frac{1}{3}D(b'bu)(bb'b'')b''_x=\frac{1}{3}D(bb''u)(bb''b')b'_x+\frac{1}{3}D(bb'u)(bb'b'')b''_x$
$=\frac{1}{3}DD'u_x+\frac{1}{3}DD'u_x=\frac{2}{3}DD'u_x$.

3. $(\beta\gamma\delta)(\gamma\delta x)\beta_u=\frac{1}{2}(\beta\gamma\delta)[(\gamma\delta x)\beta_u+(\beta\gamma x)\delta_u]=\frac{1}{2}(\beta\gamma\delta)[(\gamma\delta x)\beta_u+(\beta\gamma x)\delta_u]$
$\equiv\frac{1}{2}(\beta\gamma\delta)[(\beta\gamma\delta)x_u+(\beta\delta x)\gamma_u]=\frac{1}{2}(\beta\gamma\delta)^2x_u+\frac{1}{2}(\beta\gamma\delta)(\beta\delta x)\gamma_u$.
$\frac{1}{2}(\beta\gamma\delta)(\beta\delta x)\gamma_u=\frac{1}{2}(\widehat{a'a''\beta'\beta})(\widehat{b'b''\beta'x})\alpha_u=\frac{1}{2}(a'_{\beta'}a''_\beta-a'_\beta a''_{\beta'})(b'_{\beta'}b''_x-b'_x b''_{\beta'})\alpha_u$
$=\frac{1}{2}[a'_{\beta'}a''_\beta b'_{\beta'}b''_x\alpha_u-a'_{\beta'}a''_\beta b'_x b''_{\beta'}\alpha_u-a'_\beta a''_{\beta'}b'_{\beta'}b''_x\alpha_u+a'_\beta a''_{\beta'}b'_x b''_{\beta'}\alpha_u]$
$=2a'_{\beta'}a''_\beta b'_{\beta'}b''_x\alpha_u=2(b'b'''b^{IV})(b'''b^{IV}a')b'_{(b''a'')}b''_x\alpha_u=\frac{2}{3}D'(a'b''a'')b''_x\alpha_u$
$=-\frac{2}{3}D'b''_a b''_x\alpha_u=-\frac{2}{3}D'C_3$.

$\therefore(\beta\gamma\delta)(\gamma\delta x)\beta_u=\frac{2}{3}D'A_{112}u_x-\frac{2}{3}D'C_3$. $k_3=a_\beta b_\alpha(abu)(\alpha\beta x)=\frac{1}{3}DD'u_x$
$+A_{112}C_2-A_{112}A_{122}u_x+A_{122}C_3$.

$\therefore C'_3=\frac{2}{3}DA_{122}u_x-\frac{2}{3}DC_2+2k\{\frac{1}{3}DD'u_x-A_{112}C_2+A_{112}A_{122}u_x-A_{122}C_3\}$
$+k^2(\frac{2}{3}D'A_{112}u_x-\frac{2}{3}D'C_3)$.

## *The Form $C'_7$.*

$C'_7=\{(a_x^2+kb_x^2)(a_u^2+2k(ab)_u^2+k^2\beta_u^2),(\alpha\beta)_x^2\}_{10}^{01}=\{a_x^2\alpha_u^2,(\alpha'\beta)_x^2\}_{10}^{01}$
$+2k\{a_x^2(a'''b)_u^2,(\alpha\beta)_x^2\}_{10}^{01}+k^2\{a_x^2\beta_u^2,(\alpha\beta')_x^2\}_{10}^{01}+k\{b_x^2\alpha_u^2,(\alpha'\beta)_x^2\}_{10}^{01}$
$+2k^2\{b_x^2(ab''')_u^2,(\alpha\beta)_x^2\}_{10}^{01}+k^2\{b_x^2\beta_u^2,(\alpha\beta')_x^2\}_{10}^{01}$.

1. $\{a_x^2\alpha_u^2,(\alpha'\beta)_x^2\}_{10}^{01}=(\alpha\alpha'\beta)(\widehat{\alpha\alpha'\beta}u)a_x\alpha_u=(\alpha\alpha'\beta)\{a_\beta\alpha'_u-a_{\alpha'}\beta_u\}a_x\alpha_u$
$=-(\alpha\alpha'\beta)a_{\alpha'}\beta_u a_x\alpha_u=-(aa'''a^{IV})(a'''a^{IV}\widehat{\beta\alpha})a_x\alpha_u\beta_u$
$=-\frac{1}{3}D(\beta\alpha x)\alpha_u\beta_u=\frac{1}{3}DC_4$.

2. $\{a_x^2(a'''b)_u^2,(\alpha\beta)_x^2\}_{10}^{01}=(\widehat{a'''b\alpha\beta})(\widehat{a\alpha\beta u})a_x(a'''bu)$
$=(a'''_a b_\beta-a'''_\beta b_a)(a_\beta\alpha_u-a_a\beta_u)a_x(a'''bu)=-a'''_\beta b_a a_\beta\alpha_u a_x(a'''bu)$
$+a'''_\beta b_a a_a\beta_u a_x(a'''bu)$ (Terms containing $a'''_a b_\beta$ vanish.)
$=(-A_{122}C_7+\frac{1}{2}A_{112}C_4-\frac{1}{3}DC_6)+\frac{1}{3}DC_6=\frac{1}{2}A_{112}C_4-A_{122}C_7$.

For, $a'''_\beta a_\beta a_x b_a\alpha_u(a'''bu)-a'''^2_\beta\cdot a_x b_a\alpha_u(abu)$
$=a'''_\beta a_x b_a\alpha_u\{(a'''bu)a_\beta-(abu)a'''_\beta\}\equiv a'''_\beta a_x b_a\alpha_u[(aa'''b)u_\beta-(aa'''u)b_\beta]$
$=\frac{1}{2}b_a\alpha_u u_\beta(aa'''b)[a_x a'''_\beta-a_\beta a'''_x]-(bb'b'')(b'b''a''')b_a a_x\alpha_u(aa'''u)$
$=\frac{1}{2}b_a\alpha_u u_\beta\bar{\alpha}_b(\alpha x\beta)-\frac{1}{3}D'(a'''a'a'')(a'a''u)a'''_{(ua)}a_x=\frac{1}{2}a_b\alpha_u\alpha_b\bar{\alpha}_{(x\beta)}u_\beta$.

But, $\alpha_b\alpha_u\bar{\alpha}_b\bar{\alpha}_{(x\beta)}u_\beta - a_b^2\cdot\alpha_u u_\beta(\alpha x\beta) = \bar{\alpha}_b\alpha_u u_\beta[(\bar{\alpha}_x\beta)\alpha_b - (\alpha x\beta)\bar{\alpha}_b]$

$\equiv \bar{\alpha}_b\alpha_u u_\beta[(\alpha\bar{\alpha}x)b_\beta - (\alpha\bar{\alpha}\beta)b_x] \equiv (bb'b'')(b'b''u)\bar{\alpha}_b\alpha_u(\alpha\bar{\alpha}x) - \bar{\alpha}_b\alpha_u u_\beta(\alpha\bar{\alpha}\beta)b_x$

$= -\bar{\alpha}_b\alpha_u u_\beta b_x[a'_\beta a''_a - a'_\beta a''_a] = \bar{\alpha}_{a''}\bar{\alpha}_b a''_{(ua')}a'_\beta b_x u_\beta - \bar{\alpha}_{a'}\bar{\alpha}_b a'_{(a''u)}b_x u_\beta a''_\beta$

$= \frac{1}{3}D(bua')a'_\beta b_x u_\beta - \frac{1}{3}D(ba''u)b_x a''_\beta u_\beta = \frac{2}{3}D(abu)a_\beta b_x u_\beta = \frac{2}{3}DC_6.$

$\therefore a'''_\beta a_\beta a_x b_a\alpha_u(a'''bu) = A_{122}C_7 - \frac{1}{2}A_{112}C_4 + \frac{1}{3}DC_6.$

3. $\{a_x^2\beta_u^2, (\alpha\beta')_x^2\}_{10}^{01} = (\beta\alpha\beta')\,\widehat{(a\alpha\beta'u)}\,a_x\beta_u = (\beta\alpha\beta')\{a_{\beta'}\alpha_u - a_a\beta'_u\}a_x\beta_u$

$= a_{\beta'}\alpha_u a_x\beta_u\{b'_a b''_{\beta'} - b'_{\beta'}b''_a\} - \frac{1}{3}D(\beta'\beta x)\beta'_u\beta_u = (b''b'''b^{IV})(b'''b^{IV}a)b''_{(ub')}\alpha_u a_x b'_a$

$-(b'b'''b^{IV})(b'''b^{IV}a)b'_{(b''u)}\alpha_u a_x b''_a - 0 = -\frac{2}{3}D'C_7.$

4. $\{b_x^2\alpha_u^2, (\alpha'\beta)_x^2\}_{10}^{01} = (\alpha\alpha'\beta)\,\widehat{(b\alpha'\beta u)}\,b_x\alpha_u = (\alpha\alpha'\beta)\{b_\beta\alpha'_u - \alpha'_b\beta_u\}b_x\alpha_u$

$= b_\beta\alpha'_u b_x\alpha_u(\alpha\alpha'\beta) - b_{\alpha'}\beta_u b_x\alpha_u(\alpha\alpha'\beta) = 0 - b_{\alpha'}\beta_u b_x\alpha_u\{a'_{\alpha'}a''_\beta - a'_\beta a''_{\alpha'}\} = \frac{2}{3}DC_6.$

5. $\{b_x^2(ab''')_u^2, (\alpha\beta)_x^2\}_{10}^{01} = \widehat{(ab'''\alpha\beta)}\,\widehat{(b\alpha\beta u)}\,b_x(ab'''u)$

$= (a_a b'''_\beta - a_\beta b'''_a)(b_\beta\alpha_u - b_a\beta_u)b_x(ab'''u) = 0 - 0 - a_\beta b'''_a b_\beta\alpha_u b_x(ab'''u)$

$+a_\beta b'''_a b_a\beta_u b_x(ab'''u)^* = -\frac{1}{3}D'C_7 + [A_{112}C_6 - \frac{1}{2}A_{122}C_4 + \frac{1}{3}D'C_7\}$

$= A_{112}C_6 - \frac{1}{2}A_{122}C_4$; * similar to 2.

6. $\{b_x^2\beta_u^2, (\alpha\beta')_x^2\}_{10}^{01} = (\beta\alpha\beta')\,\widehat{(b\alpha\beta'u)}\,b_x\beta_u = (b'_a b''_{\beta'} - b'_{\beta'}b''_a)(b_{\beta'}\alpha_u - b_a\beta'_u)b_x\beta_u$

$= b'_a b''_{\beta'}b_{\beta'}\alpha_u b_x\beta_u{}^{**} - b'_a b''_{\beta'}b_a\beta'_u b_x\beta_u - b'_{\beta'}b''_a b_{\beta'}\alpha_u b_x\beta_u + b'_{\beta'}b''_a b_a\beta'_u b_x\beta_u.$

$^{**} = (b''b'''b^{IV})(b'''b^{IV}b)b''_{(ub')}b'_a b_x\alpha_u = \frac{1}{3}D'(bub')b'_a b_x\alpha_u$

$= \frac{1}{6}D'(bub')\alpha_u[b'_a b_x - b'_x b_a] = \frac{1}{6}D'(b'bu)\alpha_u\,\widehat{(b'ba x)}$

$= \frac{1}{6}D'\beta'''_u\alpha_u(\beta'''\alpha x) = -\frac{1}{6}D'C_4.$

Also, $b'_{\beta'}b''_a b_{\beta'}\alpha_u b_x\beta_u = {}^{**}$ by $b'\sim b''$, and $-b'_a b''_{\beta'}b_a\beta'_u b_x\beta_u = b'_{\beta'}b''_a b_a\beta'_u b_x\beta_u = 0.$

Hence $\{b_x^2\beta_u^2, (\alpha\beta')_x^2\}_{10}^{01} = -\frac{1}{3}D'C_4.$

$\therefore C'_7 = \frac{1}{3}DC_4 + k\{A_{112}C_4 - 2A_{122}C_7 + \frac{2}{3}DC_6\} + k^2\{-A_{122}C_4 + A_{112}C_6 - \frac{2}{3}D'C_7\}$

$+k^3(-\frac{1}{3}D'C_4).$

*The Form G'.*

$G' = [(a_x^2 + kb_x^2)(a_u^2 + 2k(ab)_u^2 + k^2\beta_u^2), (\alpha\beta)_x^2(3a_a^2a_\beta^2\beta_u^2 + 3b_\beta^2b_a^2\alpha_u^2 - 2a_a^2b_\beta^2(ab)_u^2]_{11}^{10}$

$= 3a_a^2a_\beta^2[(a_x^2\alpha_u^2, (\alpha'\beta)_x^2\beta'^2_u)_{11}^{10} + 2k(a_x^2(a'''b)_u^2, (\alpha\beta)_x^2\beta'^2_u)_{11}^{10} + k^2(a_x^2\beta_u^2, (\alpha\beta')_x^2\beta''^2_u)_{11}^{10}$

$+k(b_x^2\alpha_u^2, (\alpha'\beta)_x^2\beta'^2_u)_{11}^{10} + 2k^2(b_x^2(ab''')_u^2, (\alpha\beta)_x^2\beta'^2_u)_{11}^{10} + k^3(b_x^2\beta_u^2, (\alpha\beta')_x^2\beta''^2_u)_{11}^{10}]$

$+3b_\beta^2b_a^2[(a_x^2\alpha_u^2, (\alpha'\beta)_x^2\alpha''^2_u)_{11}^{10} + 2k(a_x^2(a'''b)_u^2, (\alpha\beta)_x^2\alpha'^2_u)_{11}^{10}$

$+k^2(a_x^2\beta_u^2, (\alpha\beta')_x^2\alpha'^2_u)_{11}^{10} + k(b_x^2\alpha_u^2, (\alpha'\beta)_x^2\alpha''^2_u)_{11}^{10} + 2k^2(b_x^2(ab''')_u^2, (\alpha\beta)_x^2\alpha'^2_u)_{11}^{10}$

$+k^3(b_x^2\beta_u^2, (\alpha\beta')_x^2\alpha'^2_u)_{11}^{10}] - 2a_a^2b_\beta^2[(a_x^2\alpha_u^2, (\alpha'\beta)_x^2(a^Vb)_u^2)_{11}^{10}$

$+2k(a_x^2(a'''b)_u^2, (\alpha\beta)_x^2(a^{IV}b''')_u^2)_{11}^{10} + k^2(a_x^2\beta_u^2, (\alpha\beta')_x^2(a'''b)_u^2)_{11}^{10}$

$+k(b_x^2\alpha_u^2, (\alpha'\beta)_x^2(ab''')_u^2)_{11}^{10} + 2k^2(b_x^2(ab''')_u^2, (\alpha\beta)_x^2(a'''b^{IV})_u^2)_{11}^{10}$

$+k^3(b_x^2\beta_u^2, (\alpha\beta')_x^2(ab^V)_u^2)_{11}^{10}].$

$$
\begin{aligned}
=&3a_\alpha^2a_\beta^2[a_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)a_x(\alpha'\beta x)+2ka_{\beta'}(\widehat{a'''b}\alpha\beta)(\widehat{a'''b}\beta'x)a_x(\alpha\beta x)\\
&+k^2a_{\beta''}(\beta\alpha\beta')(\beta\beta''x)a_x(\alpha\beta'x)+kb_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)b_x(\alpha'\beta x)\\
&+2k^2b_{\beta'}(\widehat{ab'''}\alpha\beta)(\widehat{ab'''}\beta'x)b_x(\alpha\beta x)+k^3b_{\beta''}(\beta\alpha\beta')(\beta\beta''x)b_x(\alpha\beta'x)]\\
&+3b_\beta^2b_\alpha^2[a_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)a_x(\alpha'\beta x)+2ka_{\alpha'}(\widehat{a'''b}\alpha\beta)(\widehat{a'''b}\alpha'x)a_x(\alpha\beta x)\\
&+k^2a_{\alpha'}(\beta\alpha\beta')(\beta\alpha'x)a_x(\alpha\beta'x)+kb_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)b_x(\alpha'\beta x)\\
&+2k^2b_{\alpha'}(\widehat{ab'''}\alpha\beta)(\widehat{ab'''}\alpha'x)b_x(\alpha\beta x)+k^3b_{\alpha'}(\beta\alpha\beta')(\beta\alpha'x)b_x(\alpha\beta'x)]\\
&-2a_\alpha^2b_\beta^2[(aa^{\mathrm{V}}b)(\alpha\alpha'\beta)(\widehat{\alpha a^{\mathrm{V}}b}x)a_x(\alpha'\beta x)\\
&\qquad+2k(aa^{\mathrm{IV}}b''')(\widehat{a'''b}\alpha\beta)(\widehat{a'''ba^{\mathrm{IV}}b'''}x)a_x(\alpha\beta x)\\
&+k^2(aa'''b)(\beta\alpha\beta')(\beta\widehat{a'''b}x)a_x(\alpha\beta'x)+k(bab''')(\alpha\alpha'\beta)(\alpha\widehat{ab'''}x)b_x(\alpha'\beta x)\\
&+2k^2(ba'''b^{\mathrm{IV}})(\widehat{ab'''}\alpha\beta)(\widehat{ab'''}\widehat{a'''b^{\mathrm{IV}}}x)b_x(\alpha\beta x)\\
&\qquad+k^3(bab^{\mathrm{V}})(\beta\alpha\beta')(\beta\widehat{ab^{\mathrm{V}}}x)b_x(\alpha\beta'x)].\\
=&3D\,A_{122}[0+2k(-A_{122}G)+k^2(-\tfrac{2}{3}D'G)+k(0)+2k^2(0)+k^3(0)]\\
&+3D'A_{112}[0+2k(0)+k^2(0)+k(\tfrac{2}{3}DG)+2k^2(A_{112}G)+k^3(0)]\\
&-2DD'[-\tfrac{1}{3}DG+2k(-\tfrac{1}{2}A_{112}G)+k^2(0)+k(0)\\
&\qquad+2k^2(\tfrac{1}{2}A_{122}G)+k^3(\tfrac{1}{3}D'G)]. \qquad (3)
\end{aligned}
$$

Collecting terms,

$$G'=\{\tfrac{2}{3}D^2D'+k(4DD'A_{112}-6DA_{122}^2)+k^2(6D'A_{112}^2-4DD'A_{122})-\tfrac{2}{3}DD'^2\cdot k^3\}G.$$

The calculations are shown below:

1. $a_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)a_x(\alpha'\beta x)=\tfrac{1}{2}a_{\beta'}a_x(\alpha\alpha'\beta)[(\alpha\beta'x)\alpha'_{(\beta x)}-(\alpha'\beta'x)\alpha_{(\beta x)}]$
$\equiv\tfrac{1}{2}a_{\beta'}a_x(\alpha\alpha'\beta)(\alpha\alpha'x)(\beta'\beta x)=\tfrac{1}{4}a_x(\alpha\alpha'x)(\beta\beta'x)[(\alpha\alpha'\beta)\beta'_\alpha-(\alpha\alpha'\beta')\beta_\alpha]$
$\equiv\tfrac{1}{4}a_x(\alpha\alpha'x)(\beta\beta'x)[(\alpha'\beta\beta')a_\alpha-(\alpha\beta\beta')a_{\alpha'}]=0.$

2. $b_{\beta'}(\alpha\alpha'\beta)(\alpha\beta'x)b_x(\alpha'\beta x)=(bb'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\widehat{x\alpha})b_x(\alpha\alpha'\beta)(\alpha'\beta x)$
$=\tfrac{1}{3}D'(x\alpha x)$, etc. $=0$.

3. $a_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)a_x(\alpha'\beta x)=(aa^{\mathrm{V}}a^{\mathrm{VI}})(a^{\mathrm{V}}a^{\mathrm{VI}}\widehat{x\alpha})a_x$, etc. $=0$.

4. $b_{\alpha''}(\alpha\alpha'\beta)(\alpha\alpha''x)b_x(\alpha'\beta x)=b_{\alpha''}(\alpha\alpha'\beta)[a'_{\alpha''}a''_x-a'_xa''_{\alpha''}]b_x(\alpha'\beta x)$
$=b_{\alpha''}(\alpha\alpha'\beta)a'_{\alpha''}a''_xb_x(\alpha'\beta x)-b_{\alpha''}(\alpha\alpha'\beta)a'_xa''_{\alpha''}b_x(\alpha'\beta x)$
$=(a'a^{\mathrm{V}}a^{\mathrm{VI}})(a^{\mathrm{V}}a^{\mathrm{VI}}b)[a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'}]a''_xb_x(\alpha'\beta x)$
$-(a''a^{\mathrm{V}}a^{\mathrm{VI}})(a^{\mathrm{V}}a^{\mathrm{VI}}b)[a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'}]a'_xb_x(\alpha'\beta x)=\tfrac{1}{3}Db_{\alpha'}a''_\beta a''_xb_x(\alpha'\beta x)$
$-\tfrac{1}{3}Db_\beta a''_{\alpha'}a''_xb_x(\alpha'\beta x)-\tfrac{1}{3}Db_\beta a'_{\alpha'}a'_xb_x(\alpha'\beta x)+\tfrac{1}{3}Db_{\alpha'}a'_\beta a'_xb_x(\alpha'\beta x)$
$=\tfrac{2}{3}Db_{\alpha'}a''_\beta a''_xb_x(\alpha'\beta x)-\tfrac{2}{3}D(bb'b'')(b'b''\widehat{x\alpha'})b_xa''_xa''_{\alpha'}=\tfrac{2}{3}DG-0.$

5. $(aa^{\mathrm{v}}b)(\alpha\alpha'\beta)(\widehat{\alpha a^{\mathrm{v}}b}x)a_x(\alpha'\beta x)$

$=\{a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'}\}(\alpha'\beta x)\{a^{\mathrm{v}}_x b_\alpha-a^{\mathrm{v}}_\alpha b_x\}a_x(aa^{\mathrm{v}}b)=a'_{\alpha'}a''_\beta a^{\mathrm{v}}_x b_\alpha(\alpha'\beta x)a_x(aa^{\mathrm{v}}b)$

$-a'_{\alpha'}a''_\beta a^{\mathrm{v}}_\alpha b_x(\alpha'\beta x)a_x(aa^{\mathrm{v}}b)-a'_\beta a''_{\alpha'}a^{\mathrm{v}}_x b_\alpha(\alpha'\beta x)a_x(aa^{\mathrm{v}}b)$

$+a'_\beta a''_{\alpha'}a^{\mathrm{v}}_\alpha b_x(\alpha'\beta x)a_x(aa^{\mathrm{v}}b)$

$=0-(a'a'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}\widehat{\beta x})a'_{(a''a^{\mathrm{v}})}a''_\beta b_x a_x(aa^{\mathrm{v}}b)-0$

$+(a''a'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}\widehat{\beta x})a''_{(a^{\mathrm{v}}a')}a'_\beta b_x a_x(aa^{\mathrm{v}}b)$

$=-\frac{1}{3}D(\widehat{\beta x}a''a^{\mathrm{v}})a''_\beta b_x a_x(aa^{\mathrm{v}}b)+\frac{1}{3}D(\widehat{\beta x}a^{\mathrm{v}}a')a'_\beta b_x a_x(aa^{\mathrm{v}}b)$

$=-\frac{1}{3}D(a''_\beta a^{\mathrm{v}}_x-a''_x a^{\mathrm{v}}_\beta)a''_\beta b_x a_x(aa^{\mathrm{v}}b)+\frac{1}{3}D(a^{\mathrm{v}}_\beta a'_x-a^{\mathrm{v}}_x a'_\beta)a'_\beta b_x a_x(aa^{\mathrm{v}}b)$

$=\frac{1}{3}Da_x a''_x a''_\beta a^{\mathrm{v}}_\beta b_x(aa^{\mathrm{v}}b)+\frac{1}{3}Da^{\mathrm{v}}_\beta a'_x a'_\beta b_x a_x(aa^{\mathrm{v}}b)$

$=\frac{1}{6}Da''_x a''_\beta b_x(aa^{\mathrm{v}}b)\{a_x a^{\mathrm{v}}_\beta-a_\beta a^{\mathrm{v}}_x\}+\frac{1}{6}Da'_x a'_\beta b_x(aa^{\mathrm{v}}b)\{a^{\mathrm{v}}_\beta a_x-a^{\mathrm{v}}_x a_\beta\}$

$=\frac{1}{6}Da''_x a''_\beta b_x b_\alpha(ax\beta)-\frac{1}{6}Da'_x a'_\beta b_x b_\alpha(\alpha\beta x)=-\frac{1}{6}DG-\frac{1}{6}DG=-\frac{1}{3}DG.$

6. $(bab''')(\alpha\alpha'\beta)(\widehat{\alpha ab'''}x)b_x(\alpha'\beta x)$

$=(a'_{\alpha'}a''_\beta-a'_\beta a''_{\alpha'})(\alpha'\beta x)(a_x b'''_\alpha-a_\alpha b'''_x)b_x(bab''')$

$=a'_{\alpha'}a''_\beta a_x b'''_\alpha(\alpha'\beta x)b_x(bab''')-0-a'_\beta a''_{\alpha'}a_x b'''_\alpha(\alpha'\beta x)b_x(bab''')+0$

$=2a'_{\alpha'}a''_\beta a_x b'''_\alpha(\alpha'\beta x)b_x(bab''')=2(a'a'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}\widehat{\beta x})a'_{(a''b''')}a''_\beta a_x b_x(bab''')$

$=\frac{2}{3}D(\widehat{\beta x}a''b''')a''_\beta a_x b_x(bab''')=\frac{2}{3}D\{a''_\beta b'''_x-a''_x b'''_\beta\}a''_\beta a_x b_x(bab''')$

$=-\frac{2}{3}Da''_x b'''_\beta a''_\beta a_x b_x(bab''')=-\frac{2}{3}D(b'''b'b'')(b'b''a'')b'''_{(ba)}a''_x a_x b_x=0.$

7. $a_{\beta'}(\widehat{a'''b\alpha\beta})(\widehat{a'''b\beta'x})a_x(\alpha\beta x)=a_{\beta'}a_x(a'''_\alpha b_\beta-a'''_\beta b_\alpha)(a'''_{\beta'}b_x-a'''_x b_{\beta'})(\alpha\beta x)$

$=a_{\beta'}a_x(\alpha\beta x)[a'''_\alpha b_\beta a'''_{\beta'}b_x-a'''_\alpha b_\beta a'''_x b_{\beta'}-a'''_\beta b_\alpha a'''_{\beta'}b_x+a'''_\beta b_\alpha a'''_x b_{\beta'}]$

$=(a'''a'a'')(a'a''\widehat{\beta x})a'''_{\beta'}a'_\beta a_x b_\beta b_x-(a'''a'a'')(a'a''\widehat{\beta x})a'''_x b_\beta b_{\beta'}a_{\beta'}a_x$

$-a'''_\beta b_\alpha a'''_{\beta'}b_x a_{\beta'}a_x(\alpha\beta x)+(bb'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}a)b_\alpha a'''_x a'''_\beta a_x(\alpha\beta x)$

$=\frac{1}{3}D(\beta x\beta')a_{\beta'}a_x b_\beta b_x-0-A_{122}G+\frac{1}{3}D'(aa'a'')(a'a''\widehat{\beta x})a_x a'''_x a'''_\beta=-A_{122}G.$

8. $b_{\beta'}(\widehat{ab'''\alpha\beta})(\widehat{ab'''\beta'x})b_x(\alpha\beta x)=(bb'''b^{\mathrm{IV}})(b'''b^{\mathrm{IV}}\overline{xab'''})b_x$, etc. $=0$.

9. $a_{\alpha'}(\widehat{a'''b\alpha\beta})(\widehat{a'''b\alpha'x})a_x(\alpha\beta x)=(aa'''a^{\mathrm{IV}})(a'''a^{\mathrm{IV}}\widehat{x\beta})a_x(\beta\alpha\beta')(\alpha\beta' x)$

$=\frac{1}{3}D(x\beta x)$, etc. $=0$.

10. $b_{\alpha'}(\widehat{ab'''\alpha\beta})(\widehat{ab'''\alpha'x})b_x(\alpha\beta x)=b_{\alpha'}b_x a_\beta b'''_\alpha a_x b'''_{\alpha'}(\alpha\beta x)$

But, $b_{\alpha'}b_x b'''_\alpha b'''_{\alpha'}a_\beta a_x(\alpha\beta x)-b'''^2_{\alpha'}b_x b_\alpha a_\beta a_x(\alpha\beta x)=b'''_{\alpha'}b_x a_\beta a_x(\alpha\beta x)[b_{\alpha'}b'''_\alpha-b_\alpha b'''_{\alpha'}]$

$=b'''_{\alpha'}b_x a_\beta a_x(\alpha\beta x)(\widehat{bb'''\alpha'\alpha})=\frac{1}{2}a_x b_x a_\beta(\widehat{bb'''\alpha'\alpha})[(\alpha\beta x)a'_{b'''}-(\alpha'\beta x)a_{b'''}]$

$\equiv\frac{1}{2}a_x b_x a_\beta(\widehat{bb'''\alpha'\alpha})[(\alpha\beta\alpha')b'''_x-(\alpha x\alpha')b'''_\beta]=0.$

$\therefore b_{\alpha'}(\widehat{ab'''\alpha\beta})(\widehat{ab'''\alpha'x})b_x(\alpha\beta x)=A_{112}G.$

11. $(aa^{IV}b''')\,\widehat{(a'''b\alpha\beta)}\,\widehat{(a'''ba^{IV}b'''x)}\,a_x(\alpha\beta x)=-\tfrac{1}{2}A_{112}G.$

For, $(aa^{IV}b''')\{a'''_\alpha b_\beta - a'''_\beta b_\alpha\}(\alpha\beta x)a_x\{(a'''a^{IV}b''')b_x-(ba^{IV}b''')a'''_x\}$

$=(aa^{IV}b''')\{a'''_\alpha b_\beta a_x(\alpha\beta x)-a'''_\beta b_\alpha a_x(\alpha\beta x)\}\{(a'''a^{IV}b''')b_x-(ba^{IV}b''')a'''_x\}.$

$(aa^{IV}b''')a'''_\alpha b_\beta a_x(\alpha\beta x)(a'''a^{IV}b''')b_x=0$

and $(aa^{IV}b''')a'''_\alpha b_\beta a_x(\alpha\beta x)(ba^{IV}b''')a'''_x=0.$

But, $-(aa^{IV}b''')a'''_\beta b_\alpha a_x(\alpha\beta x)(a'''a^{IV}b''')b_x$

$=-\tfrac{1}{2}b_\alpha a_x(\alpha\beta x)b_x(a'''a^{IV}b''')[(aa^{IV}b''')a'''_\beta-(aa'''b''')a^{IV}_\beta]$

$=-\tfrac{1}{2}b_\alpha a_x(\alpha\beta x)b_x(a'''a^{IV}b''')[(aa^{IV}a''')b'''_\beta+(a^{IV}b'''a''')a_\beta]$

$=\tfrac{1}{2}(aa'''a^{IV})(a'''a^{IV}b''')a_xb_a(\alpha\beta x)b_xb'''_\beta-\tfrac{1}{2}(a'''a^{IV}ab''')^2b_\alpha a_x a_\beta b_x(\alpha\beta x)$

$=-\tfrac{1}{2}A_{112}G.$ Also $(aa^{IV}b''')a'''_\beta b_\alpha a_x(\alpha\beta x)(ba^{IV}b''')a'''_x$

$=\tfrac{1}{2}a'''_\beta a_x a'''_x(\alpha\beta x)(ba^{IV}b''')\{(aa^{IV}b''')b_\alpha-(aa^{IV}b)b'''_\alpha\}$

$=\tfrac{1}{2}a'''_\beta a_x a'''_x(\alpha\beta x)(ba^{IV}b''')\{(abb''')a^{IV}_\alpha-(a^{IV}b'''b)a_\alpha\}$

$=\tfrac{1}{2}a'''_\beta a_x a'''_x a^{IV}_\alpha(\alpha\beta x)(ba^{IV}b''')(abb''')+\tfrac{1}{2}(ba^{IV}b''')^2a_\alpha a'''_\beta a_x a'''_x(\alpha\beta x)$

$=\tfrac{1}{2}(a^Va'a'')(a'a''\widehat{\beta x})a^{IV}_{(b'''b)}a'''_\beta a_x a'''_x(abb''')=\tfrac{1}{6}D(\widehat{\beta x}b'''b)a'''_\beta a_x a'''_x(abb''')$

$=\tfrac{1}{6}D\{b'''_\beta b_x-b'''_x b_\beta\}a'''_\beta a_x a'''_x(abb''')$

$=\tfrac{1}{6}D(b'''b'b'')(b'b''a''')b'''_{(ab)}b_x a_x a'''_x-\tfrac{1}{6}D(bb'b'')(b'b''a''')b_{(b'''a)}b''_x a_x a'''_x$

$=\tfrac{1}{18}DD'(a'''ab)a_x a'''_x b_x-\tfrac{1}{18}DD'(a'''b'''a)a'''_x a_x b'''_x=0.$

12. $(ba'''b^{IV})\,\widehat{(ab'''\alpha\beta)}\,\widehat{(ab'''a'''b^{IV}x)}\,b_x(\alpha\beta x)$

$=(ba'''b^{IV})\{a_\alpha b'''_\beta-a_\beta b'''_\alpha\}\beta_{(x\alpha)}\{(aa'''b^{IV})b'''_x-(b'''a'''b^{IV})a_x\{b_x$

$=0-0-(ba'''b^{IV})a_\beta b'''_\alpha(\alpha\beta x)(aa'''b^{IV})b'''_x b_x$

$+(ba'''b^{IV})a_\beta b'''_\alpha(\alpha\beta x)(b'''a'''b^{IV})a_x b_x$

$=-\tfrac{1}{2}b_x b'''_x b'''_\alpha(\alpha\beta x)(aa'''b^{IV})\{(ba'''b^{IV})a_\beta-(bab^{IV})a'''_\beta\}$

$+\tfrac{1}{2}a_\beta(\alpha\beta x)(b'''a'''b^{IV})a_x b_x\{(ba'''b^{IV})b'''_\alpha-(ba'''b''')b^{IV}_\alpha\}$

$=-\tfrac{1}{2}b_x b'''_x b'''_\alpha(\alpha\beta x)(aa'''b^{IV})\{(ba'''a)b^{IV}_\beta+(a'''b^{IV}a)b_\beta\}$

$+\tfrac{1}{2}a_\beta(\alpha\beta x)(b'''a'''b^{IV})a_x b_x\{(bb'''b^{IV})a'''_\alpha-(a'''b'''b^{IV})b_\alpha\}$

$=0+0+0+\tfrac{1}{2}(a'''b'''b^{IV})^2a_\beta a_x b_x b_\alpha(\alpha\beta x)=\tfrac{1}{2}A_{122}G.$

13. $a_{\beta''}(\beta\alpha\beta')(\beta\beta''x)a_x(\alpha\beta'x)=a_{\beta''}\{b'_\alpha b''_{\beta'}-b'_{\beta'}b''_\alpha\}\{b'_{\beta''}b''_x-b'_x b''_{\beta''}\}(\beta'x\alpha)a_x$

$=0-a_{\beta''}a_x b'_\alpha b'_x(b''b'''b^{IV})(b'''b^{IV}\widehat{x\alpha})b''_{\beta''}$

$-a_{\beta''}a_x(b'b'''b^{IV})(b'''b^{IV}\widehat{x\alpha})b'_{\beta''}b''_\alpha b''_x+0$

$=-\tfrac{1}{3}a_{\beta''}a_x b'_\alpha b'_x D'(x\alpha\beta'')-\tfrac{1}{3}a_{\beta''}a_x b''_\alpha b''_x D'(x\alpha\beta'')=-\tfrac{2}{3}D'G.$

14. $b_{\beta''}(\beta\alpha\beta')(\beta\beta''x)b_x(\alpha\beta'x)=(bb^Vb^{VI})(b^Vb^{VI}\widehat{x\beta})b_x(\beta\alpha\beta')(\alpha\beta'x)=0.$

15. $a_{\alpha'}(\beta\alpha\beta')(\beta\alpha'x)a_x(\alpha\beta'x)=(aa'''a^{IV})(a'''a^{IV}\widehat{x\beta})a_x(\beta\alpha\beta')(\alpha\beta'x)=0.$

16. $b_{a'}(\beta\alpha\beta')(\beta\alpha' x)b_x(\alpha\beta' x) = b_{a'}\{b'_a b''_{\beta'} - b'_{\beta'} b''_a\}\{b'_{a'} b''_x - b'_x b''_{a'}\}(\beta' x\alpha) b_x$

$= 0 - (b'' b''' b^{IV})(b''' b^{IV}\widehat{x\alpha}) b''_{a'} b'''_a b'_x b_x b_{a'}$

$- (b' b''' b^{IV})(b''' b^{IV}\widehat{x\alpha}) b'_{a'} b''_a b''_x b_x b_{a'} + 0 = -\tfrac{1}{3} D'(x\alpha\alpha') b'_a b'_x b_x b_{a'}$

$-\tfrac{1}{3} D'(x\alpha\alpha') b''_a b''_x b_x b_{a'} = -\tfrac{2}{3} D'(\alpha\alpha' x) b'_a b_{a'} b'_x b_x$

$= -\tfrac{2}{3} D'(a'_{a'} a''_x - a'_x a''_{a'}) b'_a b_{a'} b_x b'_x = -\tfrac{2}{3} D'(a' a''' a^{IV})(a''' a^{IV} b) a'_{(a''b')} a''_x b_x b'_x$

$+\tfrac{2}{3} D'(a'' a''' a^{IV})(a''' a^{IV} b) a''_{(b'a')} b_x a'_x b'_x = -\tfrac{2}{9} DD'(ba'' b') b_x b'_x a''_x$

$+\tfrac{2}{9} DD'(bb'a') b_x b'_x a'_x = 0.$

17. $(aa'''b)(\beta\alpha\beta')(\beta\widehat{a'''bx}) a_x(\alpha\beta' x)$

$= (aa'''b)(\alpha\beta' x) a_x \{b'_a b''_{\beta'} - b'_{\beta'} b''_a\}\{a'''_x b_\beta - a'''_\beta b_x\}$

$= (aa'''b)(\alpha\beta' x) a_x \{b'_a b''_{\beta'} a'''_x b_\beta - b'_a b''_{\beta'} a'''_\beta b_x - b'_{\beta'} b''_a a'''_x b_\beta + b'_{\beta'} b''_a a'''_\beta b_x\}$

$= (b'' b''' b^{IV})(b''' b^{IV}\widehat{x\alpha}) b''_{(bb')} a'''_x b'_a a_x (aa'''b)$

$- (b'' b''' b^{IV})(b''' b^{IV}\widehat{x\alpha}) b''_{(a'''b')} b_x b'_a a_x (aa'''b)$

$- (b' b''' b^{IV})(b''' b^{IV}\widehat{x\alpha}) b'_{(b''b)} a'''_x b''_a a_x (aa'''b)$

$+ (b' b''' b^{IV})(b''' b^{IV}\widehat{x\alpha}) b'_{(b''a''')} b_x b''_a a_x (aa'''b)$

$= \tfrac{1}{3} D'(bb'\widehat{x\alpha}) b'_a a_x a'''_x (aa'''b) - \tfrac{1}{3} D'(a''' b'\widehat{x\alpha}) b_x b'_a a_x (aa'''b)$

$-\tfrac{1}{3} D'(b''\widehat{bx\alpha}) b''_a a_x a'''_x (aa'''b) + \tfrac{1}{3} D'(b'' a'''\widehat{x\alpha}) b_x b''_a a_x (aa'''b)$

$= -\tfrac{2}{3} D'\{a'''_x b'_a - a'''_a b'_x\} b_x b'_a a_x (aa'''b) = -\tfrac{2}{3} D' b'^2_a a'''_x b_x a_x (aa'''b)$

$+\tfrac{2}{3} D' a'''_a b'_x b_x b'_a a_x (aa'''b) = \tfrac{2}{3} D'(a''' a' a'')(a' a'' b') a'''_{(ba)} b'_x b_x a_x$

$= \tfrac{2}{9} DD'(b'ba) b'_x b_x a_x = 0.$

18. $(bab^{V})(\beta\alpha\beta')(\beta\widehat{ab^{V}x}) b_x(\alpha\beta' x)$

$= (bab^{V})(\beta\widehat{ab^{V}x}) b_x \{b'_a b''_{\beta'} - b'_{\beta'} b''_a\}\{a'_{\beta'} a''_x - a'_x a''_{\beta'}\}$

$= (bab^{V}) b_x [b'_a b''_{\beta'} a'_{\beta'} a''_x - b'_a b''_{\beta'} a'_x a''_{\beta'} - b'_{\beta'} b''_a a'_{\beta'} a''_x + b'_{\beta'} b''_a a'_x a''_{\beta'}](a_x b^{V}_\beta - a_\beta b^{V}_x)$

$= (b'' b''' b^{IV})(b''' b^{IV} a') b''_{(b^{V} b')} b'_a a''_x a_x (bab^{V}) b_x$

$- (b'' b''' b^{IV})(b''' b^{IV} a') b''_{(ab')} b'_a a''_x b^{V}_x (bab^{V}) b_x$, etc.

$= \tfrac{1}{3} D'(a' b^{V} b') b'_a a''_x a_x b_x (bab^{V}) - \tfrac{1}{3} D'(a' ab') b'_a a''_x b^{V}_x b_x (bab^{V})$, etc.

$= \tfrac{1}{6} D' a''_x a_x b_x (a' b^{V} b')[(bab^{V}) b'_a - (bab') b^{V}_a] - 0$, etc.

$= \tfrac{1}{6} D' a''_x a_x b_x (a' b^{V} b')[(ab^{V} b') b_a - (bb^{V} b') a_a]$, etc.

$= \tfrac{1}{6} D' a''_x a_x b_x (a' b^{V} b')(ab^{V} b') b_a$, etc. $= \tfrac{1}{6} D' a''_x a_x b_x a'_\beta a_\beta b_a$, etc.

$= \tfrac{1}{12} D' a_x b_x a_\beta b_a [a''_x a'_\beta - a'_x a''_\beta]$, etc. $= \tfrac{1}{12} D' a_x b_x a_\beta b_a (\alpha\beta x)$, etc.

$= \tfrac{1}{12} D' G + \tfrac{1}{12} D' G + \tfrac{1}{12} D' G + \tfrac{1}{12} D' G = \tfrac{1}{3} D' G.$

Combining these results as in (3) we obtain $G'$.

## TABLE II.

*System of* $a_x^2+kb_x^2$, $(\alpha\beta x)^2$.

1. $K = a_x^2+kb_x^2$.
2. $F = (\alpha\beta x)^2$.
3. $(L)' = \alpha_u^2+2k(abu)^2+k^2\beta_u^2$.
4. $(L')' = \frac{4}{9}\{3DA_{122}u_\beta^2+3D'A_{112}u_a^2-2DD'(abu)^2\}$.
5. $(D)' = D+3kA_{112}+3k^2A_{122}+k^3D'$.
6. $(D')' = \frac{8}{27}\{9A_{112}A_{122}-DD'\}DD'$.
7. $\Phi' = \frac{1}{3}DL'+A_{122}L+k\{\frac{1}{3}D'L+A_{112}L'\}$.
8. $C_1' = C_8-kC_5$.
9. $A_{122}' = \frac{4}{9}\{3DA_{122}^2+DD'A_{112}+k(3D'A_{112}^2+DD'A_{122})\}$.
10. $C_2' = {}_4\{3DA_{122}C_2+DD'C_3+k(3D'A_{112}C_3+DD'C_2)\}$.
11. $C_3' = (\frac{2}{3}DA_{122}u_x-\frac{2}{3}DC_2)+2k(\frac{1}{3}DD'u_x-A_{112}C_2+A_{112}A_{122}u_x-A_{122}C_8)$
$+k^2(\frac{2}{3}D'A_{112}u_x-\frac{2}{3}D'C_8)$.
12. $A_{112}' = \frac{4}{9}DA_{122}+2k(\frac{1}{3}DD'+A_{112}A_{122})+k^2\cdot\frac{4}{9}D'A_{112}$.
13. $C_4' = \frac{4}{9}[3DA_{122}C_4-2DD'C_7+6k(DA_{122}C_6-D'A_{112}C_7)$
$-k^2(3D'A_{112}C_4-2DD'C_6)]$.
14. $F' = \frac{4}{9}\{3DA_{122}F+2DD'A_{112}f-\frac{2}{3}D^2D'g$
$+2k(3DA_{122}^2g-DD'A_{122}f-A_{112}DD'g+3A_{112}^2D'f+DD'F)$
$+k^2(3D'A_{112}F+2DD'A_{122}g-\frac{2}{3}DD'^2f)\}$.
15. $C_5' = \frac{4}{9}\{2DD'A_{112}C_8+\frac{2}{3}D^2D'C_5$
$+2k(3D'A_{112}^2C_8+DD'A_{112}C_5-3DA_{122}^2C_5-DD'A_{122}C_8)$
$+k^2(-2DD'A_{122}C_5-\frac{2}{3}DD'^2C_8)\}$.
16. $C_6' = \frac{4}{9}\{3DA_{122}^2C_4-2DD'A_{122}C_7-DD'A_{112}C_4+\frac{2}{3}D^2D'C_6$
$+k(-3D'A_{112}^2C_4+2DD'A_{112}C_6+DD'A_{122}C_4-\frac{2}{3}DD'^2C_7)\}$.
17. $C_7' = \frac{1}{3}DC_4+k(A_{112}C_4-2A_{122}C_7+\frac{2}{3}DC_6)$
$+k^2(-A_{122}C_4+A_{112}C_6-\frac{2}{3}D'C_7)+k^3(-\frac{1}{3}D'C_4)$.
18. $C_8' = \frac{4}{9}\{3DA_{122}C_8-\frac{2}{3}D^2D'C_1+k(6DA_{122}^2C_1-4DD'A_{112}C_1+2DD'C_8-3DA_{122}C_5)$
$+k^2(-6D'A_{112}^2C_1+4DD'A_{122}C_1-2DD'C_5+3D'A_{112}C_8)$
$+k^3(\frac{2}{3}DD'^2C_1-3D'A_{112}C_5)\}$.
19. $G' = \frac{4}{9}\{\frac{2}{3}D^2D'+k(4DD'A_{112}-6DA_{122}^2)$
$+k^2(6D'A_{112}^2-4DD'A_{122})-\frac{2}{3}DD'^2\cdot k^3\{G$.
20. $\Gamma' = \frac{4}{9}\{\frac{2}{3}D^2D'+k(4DD'A_{112}-6DA_{122}^2)$
$+k^2(6D'A_{112}^2-4DD'A_{122})-\frac{2}{3}DD'^2\cdot k^3\{\Gamma_8$

## § 3. SYSTEM OF $a_x^2+kb_x^2,\ c_x^2$.

### TABLE III.

*$(a_x^2+kb_x^2, c_x^2)_{kl}^{ij}$ in terms of the system of Baker.*

$$\Phi'' = (a_x^2+kb_x^2, c_x^2)_{00}^{02} = \Phi_2+k\Phi_1.$$

$$C_1'' = (a_x^2+kb_x^2, c_x^2)_{00}^{01} = C_{12}+kC_{11}.$$

$$A_{122}'' = (a_x^2+kb_x^2, \gamma_u^2)_{00}^{20} = A_1+kA_2.$$

$$C_2'' = (a_x^2+kb_x^2, \gamma_u^2)_{00}^{10} = C_{21}+kC_{22}.$$

$$C_3'' = [\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2, c_x^2]_{10}^{00} = C_{23}+2kM+k^2C_{24}.$$

$$A_{112}'' = [\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2, c_x^2]_{20}^{00} = A_3+2kN+k^2A_4.$$

$$C_4'' = [\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2, \gamma_u^2]_{01}^{00} = C_{42}+2k(O_1-O_2)+k^2C_{41}.$$

$$F'' = [\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2, \gamma_u^2]_{02}^{00} = F_2+2k(A_1g-2P+A_2f)+k^2F_1.$$

$$C_5'' = [\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2, c_x^2\gamma_u^2]_{11}^{00} = C_{51}+2k(Q_1-Q_2)+k^2C_{52}.$$

$$C_6'' = (a_x^2+kb_x^2, c_x^2\gamma_u^2)_{00}^{11} = C_{61}+kC_{62}.$$

$$C_7'' = [(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2), c_x^2]_{10}^{01}$$
$$= C_{63}+k(2R_1-S_1J+2O_3)+k^2(2R_2-S_2J+2O_4)+k^3C_{64}.$$

$$C_8'' = [(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2), \gamma_u^2]_{01}^{10}$$
$$= C_{53}+k(2A_1C_1-2fS_3+3T_1-T_2-JU_1)$$
$$+k^2(-2A_2C_1+2gS_3+3T_3-T_4-JU_2)+k^3C_{54}.$$

$$G'' = [(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2), c_x^2\gamma_u^2]_{11}^{10}$$
$$= G_1+k[2A_1V-2fW+X_1]+k^2[-2A_2V+2gW+X_2]+k^3G_2.$$

$$\Gamma'' = [(a_x^2+kb_x^2)(\alpha_u^2+2k(ab)_u^2+k^2\beta_u^2), c_x^2\gamma_u^2]_{10}^{11}$$
$$= \Gamma_2+k[2A_1Y+S_1L''-2S_3\Phi_2+Z_1]+k^2[-2A_2Y-S_2L''+2S_3\Phi_1+Z_2]+k^3\Gamma_1.$$

# *Note on Seminvariants of Systems of Partial Differential Equations.*

By A. L. Nelson.

## 1. *Introduction.*

In the discussion of the projective differential geometry of a geometric configuration by means of Wilczynski's method, one of the necessary steps is the construction of a fundamental set of seminvariants, that is, such a set that any seminvariant whatever may be expressed in terms of them and their derivatives. In the cases where the completely integrable system of partial differential equations employed has one dependent variable, this construction has been accomplished by the reduction of the system of equations to its canonical form, the independent coefficients of which form are the fundamental set required. Let us illustrate the process by the case of plane nets.* The completely integrable system of equations for this case is the following:

$$y_{uu}=ay_u+by_v+cy, \quad y_{uv}=a'y_u+b'y_v+c'y, \quad y_{vv}=a''y_u+b''y_v+c''y. \tag{1}$$

The transformation

$$y=\lambda\bar{y} \tag{2}$$

yields a new system of equations of the same form as (1), with the coefficients

$$\left.\begin{aligned}
&\bar{a} = a - 2\frac{\lambda_u}{\lambda}, \quad \bar{b} = b, && \bar{c} = c + a\frac{\lambda_u}{\lambda} + b\frac{\lambda_v}{\lambda} - \frac{\lambda_{uu}}{\lambda},\\
&\bar{a}' = a' - \frac{\lambda_v}{\lambda}, \quad \bar{b}' = b' - \frac{\lambda_u}{\lambda}, && \bar{c}' = c' + a'\frac{\lambda_u}{\lambda} + b'\frac{\lambda_v}{\lambda} - \frac{\lambda_{uv}}{\lambda},\\
&\bar{a}'' = a'', && \bar{b}'' = b'' - 2\frac{\lambda_v}{\lambda}, \quad \bar{c}'' = c'' + a''\frac{\lambda_u}{\lambda} + b''\frac{\lambda_v}{\lambda} - \frac{\lambda_{vv}}{\lambda}.
\end{aligned}\right\} \tag{3}$$

The integrability conditions enable us to find a function $p(u, v)$, such that $p_u=a+b'$, $p_v=a'+b''$. We find also that under the transformation (2) these combinations become

$$\bar{p}_u=p_u-3\frac{\lambda_u}{\lambda}, \quad \bar{p}=p_v-3\frac{\lambda_v}{\lambda}.$$

* Cf. E. J. Wilczynski, "One-Parameter Families and Nets of Plane Curves," *Trans. Amer. Math. Soc.*, Vol. XII (1911), pp. 473–510.

If, therefore, $\lambda$ be so chosen that

$$\frac{\lambda_u}{\lambda}=\tfrac{1}{3}p_u, \quad \frac{\lambda_v}{\lambda}=\tfrac{1}{3}p_v,$$

we get the special unique form of (1), which shall be indicated by capital letter coefficients, characterized by the relations $A+B'=0$, $A'+B''=0$. This is the canonical form of (1). We list its coefficients for later comparison.

$$\left.\begin{aligned}
A &= \tfrac{1}{3}(a-2b'), \quad B = b, \quad C = c+\tfrac{2}{9}a^2+\tfrac{1}{3}ab'+\tfrac{1}{3}a'b+\tfrac{1}{3}bb''\\
&\qquad -\tfrac{1}{3}a_u-\tfrac{1}{3}b'_u-\tfrac{1}{9}b'^2,\\
A' &= \tfrac{1}{3}(2a'-b''), \quad B' = \tfrac{1}{3}(2b'-a), \quad C' = c'+\tfrac{2}{9}aa'+\tfrac{5}{9}a'b'+\tfrac{2}{9}b'b''\\
&\qquad -\tfrac{1}{9}ab''-\tfrac{1}{3}a_v-\tfrac{1}{3}b'_v,\\
A'' &= a'', \quad B'' = \tfrac{1}{3}(b''-2a'), \quad C'' = c''+\tfrac{2}{9}b''^2+\tfrac{1}{9}a'b''+\tfrac{1}{3}a''b'+\tfrac{1}{3}aa''\\
&\qquad -\tfrac{1}{3}b''_v-\tfrac{1}{3}a'_v-\tfrac{1}{9}a'^2.
\end{aligned}\right\} \quad (4)$$

Only seven of these coefficients are independent. They are the fundamental seminvariants. For, consider any system of equations of the form (1), to which we shall refer as $(s)$, and any other system $(t)$, obtained from it by any transformation of the type (2). Let $(s)$ be reduced to its canonical form $(S)$, and $(t)$ to its canonical form $(T)$. The coefficients of $(S)$ are the same combinations of the coefficients of $(s)$ as the corresponding coefficients of $(T)$ are of those of $(t)$. But, since the canonical form is unique, the corresponding coefficients of $(S)$ and $(T)$ are identical. Hence they are seminvariants. Moreover, any seminvariant whatever, $I$, is a function of the original coefficients of $(s)$, *i. e.*, $I=I(a, b, \ldots, c'', a_u, a_v, \ldots)$. Since it is a seminvariant, it must be identical with the same function of the corresponding coefficients of any system of equations obtained from $(s)$ by a transformation (2). In particular, $I=I(A, B, \ldots, C'', A_u, A_v, \ldots)$. Therefore $I$ is a function of the seven independent coefficients of the canonical form and of their derivatives. Hence these coefficients are a fundamental set of seminvariants.

The determination of this canonical form depends upon the ability to find such a function $p$. Green* has shown that this can be done for completely integrable systems with one dependent variable and $n$ independent variables, provided certain conditions are fulfilled. However, the expressions for $p_u$ and $p_v$, which, with their derivatives, must occur in the coefficients of the canonical

*G. M. Green, "The Linear Dependence of Functions of Several Variables, and Completely Integrable Systems of Homogeneous Linear Partial Differential Equations," *Trans. Amer. Math. Soc.*, Vol. XVII (1916), pp. 483–516.

form, are not always simple, and as a result the seminvariants which come from this regulation method are sometimes rather complicated when expressed in terms of the original coefficients. We proceed to indicate how (in most cases, at least) an alternative set of simpler seminvariants may be computed.

2. *The Effect of Transformation* (2) *in the General Case.*

We assume that the completely integrable system of equations (corresponding to (1)), to which we shall refer as (1′), has one dependent variable, $y$,* and $n$ independent variables, $u_1, \ldots, u_n$, $(n>1)$. It will then consist of a certain number, $p$, of equations which express certain $p$ derivatives of $y$ in terms of $q$ chosen derivatives (which Green has called the primary derivatives). Let us also assume that among the $q$ primary derivatives are none of order higher than the order of the lowest of the $p$ derivatives in the left members of the $p$ equations.

The transformation (2) gives rise to the following expressions of the various derivatives of $y$ in terms of the derivatives of $\bar{y}$.

$$\left.\begin{aligned} y &= \lambda \bar{y}, \\ y_{u_i} &= \lambda \bar{y}_{u_i} + \lambda_{u_i} \bar{y}, \quad (i=1, 2, \ldots, n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ y_{u_1 \ldots u_n}^{l_1 \ldots l_n} &= \sum_{\substack{p_1=0, \ldots, l_1 \\ \cdots\cdots \\ p_n=0, \ldots, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} \lambda_{u_1 \ldots u_n}^{p_1 \ldots p_n} \bar{y}_{u_1 \ldots u_n}^{l_1 - p_1 \ldots l_n - p_n}, \end{aligned}\right\} \tag{2′}$$

where

$$\binom{l_j}{p_j} \equiv \frac{l_j!}{p_j!\,(l_j - p_j)!} \quad \text{and} \quad y_{u_1 \ldots u_n}^{l_1 \ldots l_n} \equiv \frac{\partial^{l_1 + \cdots + l_n} y}{\partial u_1^{l_1} \ldots \partial u_n^{l_n}}.$$

If we substitute (2′) in the system (1′), and arrange the resulting equations properly, we get a new system of the same form as (1′), in which the dependent variable is $\bar{y}$. The coefficients of this new system, which, in collected form, we shall refer to as (3′), are equal to the corresponding coefficients of (1′) plus linear combinations of fractions whose numerators are derivatives of $\lambda$, and whose common denominator is $\lambda$. Each coefficient (3′) receives additional terms from those $y$-derivatives in its equation which come by differentiation from the $y$-derivative belonging to this coefficient. (Thus, for example, in (3), $\bar{b}'$, the coefficient of $\bar{y}_v$, can receive only one additional term, $-\frac{\lambda_u}{\lambda}$, which comes from $y_{uv}$; $\bar{c}'$ receives one $\lambda$-fraction from each of the derivatives $y_u$, $y_v$, $y_{uv}$.)

* This assumption is merely for the sake of simplifying the discussion. The process may be readily extended to cover the cases where the number of dependent variables is greater than one.

We arrange these coefficients (3′) in classes as follows: In the zero-th class shall be placed those coefficients in each row which are coefficients of those primary derivatives having the same order as the derivative in the left member of the equation belonging to that row. The $i$-th class shall consist of the coefficients of those primary derivatives in each row of order $i$ less than the derivative in the left member of the equation belonging to that row. (Thus, in table (3), the coefficients in the first and second columns are of the first class; those in the third column are of the second class.) The zero-th class coefficients, if present, must be seminvariants, since their $y$-derivatives can not yield by differentiation any other $y$-derivative in their equations. The first class coefficients will be altered, if at all, by the addition of multiples of $\lambda$-fractions of the first order, the coefficients of which $\lambda$-fractions are integers or integral multiples of zero-th class coefficients (3′). In general, the $i$-th class coefficients will be altered by the addition of linear combinations of $\lambda$-fractions of order not higher than $i$. Each one of the coefficients of these $\lambda$-fractions will be an integer or an integral multiple of some one coefficient (3′) of classes 0 to $i-1$.

### 3. *Pseudo-Canonical Forms of* (1′).

Now it very frequently happens that *under suitable restrictions* we may make $n$ first class coefficients (3′) vanish by a proper choice of $\lambda$. For example, in (3), under the assumption $a'_u=b'_v$, we might have made $\overline{a}'=\overline{b}'=0$. Or we might have made $\overline{a}=\overline{b}''=0$, provided $a_v=b''_u$. If we can do this, under the necessary restriction the new form of the completely integrable system determined by this choice of $\lambda$ is unique, because of the nature of the equations in $\lambda$. We shall call this unique form a *pseudo-canonical form of* (1′). Moreover, the necessary restrictions are always seminvariantive ones. For, consider certain $n$ such first class coefficients,

$$\bar{\alpha}_i=\alpha_i-k_i\frac{\partial}{\partial u_i}\log\lambda, \qquad (i=1, 2, \ldots, n), \tag{5}$$

where the $k_i$ are at worst integral multiples of seminvariant coefficients of (1′). In order to choose $\lambda$ to satisfy the equations

$$\alpha_i-k_i\frac{\partial}{\partial u_i}\log\lambda=0, \qquad (i=1, 2, \ldots, n), \tag{6}$$

the assumptions necessary are

$$\frac{\partial}{\partial u_i}\left(\frac{\alpha_j}{k_j}\right)=\frac{\partial}{\partial u_j}\left(\frac{\alpha_i}{k_i}\right), \qquad (i, j=1, 2, \ldots, n), \tag{7}$$

which are easily seen to be seminvariantive from (5).

Let us assume the necessary restrictions (7) and choose $\lambda$ in accordance with (6). However, there is a precaution to be observed in substituting for $\lambda$. Whenever a cross-derivative,

$$\frac{1}{\lambda} \cdot \frac{\partial^{p_1+\ldots+p_n}\lambda}{\partial u_1^{p_1} \ldots . \partial u_n^{p_n}} \quad \text{(at least two } p\text{'s} \neq 0),$$

occurs, equations (7) afford us a choice in the substitution. We must, throughout a coefficient (3′), replace such cross-derivatives by the proper derivatives of the *same* members of equations (7). The reason for this precaution will become apparent in the next section. Then, since the restrictions are seminvariantive, and since the pseudo-canonical form is unique, the coefficients of this form are seminvariants, by precisely the same argument as was made in Section 1 for the coefficients of the canonical form in the case of plane nets.

## 4. *The Coefficients of a Pseudo-Canonical Form are a Fundamental Set of Seminvariants.*

Now, if the restrictions (7) be removed, the pseudo-canonical form will no longer exist, *but the combinations we have obtained are still seminvariants.* For, let us form the same combinations of the coefficients (3′). If we do this under the assumption (7), of course all traces of $\lambda$ will disappear. That this must also be true without these assumptions is evident from the following considerations:

Let any one of these combinations be denoted by $f(a, b, \ldots.)$. It is composed of coefficients of (1′) of various classes. However, from the nature of the choice of $\lambda$, the only derivatives which appear in $f(a, b, \ldots.)$ are derivatives of coefficients of the first class. Now, let us form the same combination of the coefficients (3′), $f(\bar{a}, \bar{b}, \ldots.)$, and examine the resulting expression. It may acquire derivatives of coefficients of (1′) only from the following sources: 1. Those derivatives which appeared in $f(a, b, \ldots.)$ will remain. These, as we have noted, are derivatives of coefficients of (1′) of the first class only. 2. Those arguments, $\bar{a}, \bar{b}, \ldots.$, which are of the first class (whose expressions are given by (3′)), may have as coefficients of $\lambda$-fractions zero-th class coefficients of (1′). Hence, when the expressions (3′) for these arguments are substituted in $f(\bar{a}, \bar{b}, \ldots.)$, the resulting expression may contain derivatives of these zero-th class coefficients. It thus appears that the only derivatives of coefficients of (1′) of any class except the zero-th which

may occur are those from the first source. Moreover, we have taken the precaution that among these derivatives in $f(a, b, \ldots.)$ shall appear only one member of any of equations (7) and its derivatives. Hence, if we collect the expression resulting from substitution of the values (3′) for the arguments in $f(\bar{a}, \bar{b}, \ldots.)$, the collection being with respect to the $\lambda$-fractions of various orders, no one of these fractions may have as coefficient an expression which vanishes by virtue of assumptions (7). Therefore, since in the resulting expression all terms in $\lambda$ cancelled under assumptions (7), the same thing must happen without these assumptions.

It is to be observed that in a pseudo-canonical form, there are $pq-n$ non-vanishing coefficients, precisely the number of independent coefficients in Green's generalized canonical form. Moreover, these coefficients are independent, since no two arise from the same coefficient of the original system (1′). From this it may be argued that the seminvariant coefficients of a pseudo-canonical form are a fundamental set.

### 5.* *Seminvariants Arising from Cross-Derivatives of log $\lambda$.*

In certain cases there is a lack of definiteness in the formation of seminvariants by the process outlined above. As mentioned previously, whenever a $\lambda$-fraction involving a cross-derivative of $\lambda$ occurs in a coefficient (3′), either member of the proper equation of (7) may be used in forming the seminvariant corresponding to this coefficient. Indeed, certain expressions of the type

$$\frac{m\dfrac{\partial}{\partial u_i}\left(\dfrac{\alpha_j}{k_j}\right)+n\dfrac{\partial}{\partial u_j}\left(\dfrac{\alpha_i}{k_i}\right)}{m+n}$$

may be used instead of $\dfrac{\partial}{\partial u_i}\left(\dfrac{\alpha_j}{k_j}\right)$ or $\dfrac{\partial}{\partial u_j}\left(\dfrac{\alpha_i}{k_i}\right)$.

A coefficient of the type in question may be put in the form

$$a_0+\sum_{\substack{i,j=1,\ldots,n\\ i\leq j}} a^{(i,j)}D^{(i,j)}+\text{sum of terms of type } aD^{(i,j)}_{u_1^{l_1}\ldots u_n^{l_n}},(l_1,\ldots,l_n=0,1,\ldots.),$$

where $D^{(i,j)}\equiv\dfrac{\partial^2}{\partial u_i\partial u_j}\log\lambda$. Let us denote one of the possible substitutions for $D^{(i,j)}$ by $[D^{(i,j)}]$, and the other by $\{D^{(i,j)}\}$. We wish to find what expressions of the form

$$F^{(i,j)}\equiv\frac{m^{(i,j)}[D^{(i,j)}]+n^{(i,j)}\{D^{(i,j)}\}}{m^{(i,j)}+n^{(i,j)}},\qquad (m^{(i,j)}+n^{(i,j)}\neq 0), \tag{8}$$

* Section 5 was added at Professor Wilczynski's suggestion.

may be used to replace $D^{(i,j)}$. We shall find it convenient to use the notation

$$(M^{(i,j)}_{\substack{p_1 \\ u_1 \ldots u_n}} {}^{p_n} D^{(i,j)}_{\substack{q_1 \\ u_1 \ldots u_n}} {}^{q_n}) \equiv m^{(i,j)}_{\substack{p_1 \\ u_1 \ldots u_n}} {}^{p_n} [D^{(i,j)}_{\substack{q_1 \\ u_1 \ldots u_n}} {}^{q_n}] + n^{(i,j)}_{\substack{p_1 \\ u_1 \ldots u_n}} {}^{p_n} \{D^{(i,j)}_{\substack{q_1 \\ u_1 \ldots u_n}} {}^{q_n}\}, \tag{9}$$

and it is easily verified that

$$(M^{(i,j)} D^{(i,j)})_{u_k} = (M^{(i,j)} D^{(i,j)}_{u_k}) + (M^{(i,j)}_{u_k} D^{(i,j)}), \text{ etc.,} \quad (i, j, k = 1, \ldots, n).$$

Now, if we denote by $\bar{\theta} = \theta + \delta\theta$ the same combination of the transformed coefficients (3′) as $\theta$ is of the original coefficients of (1′), we see from (5) that $\overline{[D^{(i,j)}]} = [D^{(i,j)}] - D^{(i,j)}$, $\overline{\{D^{(i,j)}\}} = \{D^{(i,j)}\} - D^{(i,j)}$. If, therefore, $F^{(i,j)}$ is to replace $D^{(i,j)}$, it must happen that

$$\delta F^{(i,j)}_{\substack{l_1 \\ u_1 \ldots u_n}} {}^{l_n} + D^{(i,j)}_{\substack{l_1 \\ u_1 \ldots u_n}} {}^{l_n} = 0, \quad (l_1, \ldots, l_n = 0, 1, \ldots).$$

In order to compute $\delta F^{(i,j)}_{\substack{l_1 \\ u_1 \ldots u_n}} {}^{l_n}$, let us successively differentiate the equations

$$(m^{(i,j)} + n^{(i,j)}) F^{(i,j)} = (M^{(i,j)} D^{(i,j)}),$$

which result from (8) and (9). We obtain

$$\sum_{\substack{p_1=0, \ldots, l_1 \\ \cdots\cdots\cdots \\ p_n=0, \ldots, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (m^{(i,j)} + n^{(i,j)}) \, {}^{p_1 \ldots p_n}_{u_1 \ldots u_n} F^{(i,j)}_{\substack{l_1 - p_1 \\ u_1 \ldots u_n}} {}^{l_n - p_n}$$

$$= \sum_{\substack{p_1=0, \ldots, l_1 \\ \cdots\cdots\cdots \\ p_n=0, \ldots, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (M^{(i,j)}_{\substack{p_1 \\ u_1 \ldots u_n}} {}^{p_n} D^{(i,j)}_{\substack{l_1 - p_1 \\ u_1 \ldots u_n}} {}^{l_n - p_n}).$$

From these equations

$$F^{(i,j)}_{\substack{l_1 \\ u_1 \ldots u_n}} {}^{l_n} = \frac{1}{m^{(i,j)} + n^{(i,j)}} \left\{ \sum_{\substack{p_1=0, \ldots, l_1 \\ \cdots\cdots\cdots \\ p_n=0, \ldots, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (M^{(i,j)}_{\substack{p_1 \\ u_1 \ldots u_n}} {}^{p_n} D^{(i,j)}_{\substack{l_1 - p_1 \\ u_1 \ldots u_n}} {}^{l_n - p_n}) \right.$$

$$- \sum_{s=1}^{n} \sum_{r \neq s}^{p_r=1, \ldots, l_r} \binom{l_1}{p_1} \cdots \binom{l_{s-1}}{p_{s-1}} \binom{l_{s+1}}{p_{s+1}} \cdots \binom{l_n}{p_n} (m^{(i,j)} + n^{(i,j)}) \, {}^{p_1 \ldots p_{s-1} \; p_{s+1} \ldots p_n}_{u_1 \ldots u_{s-1} \; u_{s+1} \ldots u_n} F^{(i,j)}_{\substack{l_1 - p_1 \; l_s \; l_n - p_n \\ u_1 \ldots u_s \ldots u_n}}$$

$$\left. - \sum_{\substack{p_1=1, \ldots, l_1 \\ \cdots\cdots\cdots \\ p_n=1, \ldots, l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} (m^{(i,j)} + n^{(i,j)}) \, {}^{p_1 \ldots p_n}_{u_1 \ldots u_n} F^{(i,j)}_{\substack{l_1 - p_1 \ldots l_n - p_n \\ u_1 \ldots u_n}} \right\},$$

and $\overline{F^{(i,j)}_{\substack{l_1 \\ u_1 \ldots u_n}} {}^{l_n}}$ will have a similar expression with $\overline{m^{(i,j)}}$, $\overline{n^{(i,j)}}$, $\overline{M^{(i,j)}}$, $\overline{D^{(i,j)}}$ and $\overline{F^{(i,j)}}$, replacing $m^{(i,j)}$, $n^{(i,j)}$, $M^{(i,j)}$, $D^{(i,j)}$ and $F^{(i,j)}$, respectively. Use of the relation

$$\delta F^{(i,j)}_{\substack{l_1 \\ u_1 \ldots u_n}} {}^{l_n} = \overline{F^{(i,j)}_{\substack{l_1 \\ u_1 \ldots u_n}} {}^{l_n}} - F^{(i,j)}_{\substack{l_1 \\ u_1 \ldots u_n}} {}^{l_n}$$

gives us, after considerable labor,

$$
\left.
\begin{aligned}
&\delta F^{(i,j)}_{\substack{l_1 \\ u_1 \dots u_n}} + D^{(i,j)}_{\substack{l_1 \\ u_1 \dots u_n}} = \\
&\text{(a)} \quad \sum_{\substack{p_1=0,\dots,l_1 \\ \dots\dots\dots \\ p_n=0,\dots,l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} \left( \delta \frac{m^{(i,j)}_{\substack{p_1 \\ u_1 \dots u_n}}}{m^{(i,j)} + n^{(i,j)}} \left[ D^{(i,j)}_{\substack{l_1-p_1 \\ u_1 \dots u_n}} \right]^{l_n-p_n} \right. \\
&\qquad \left. + \delta \frac{n^{(i,j)}_{\substack{p_1 \\ u_1 \dots u_n}}}{m^{(i,j)} + n^{(i,j)}} \left\{ D^{(i,j)}_{\substack{l_1-p_1 \\ u_1 \dots u_n}} \right\}^{l_n-p_n} \right) \\
&\text{(b)} \quad - \sum_{s=1}^{n} \sum_{r \neq s}^{p_r=1,\dots,l_r} \binom{l_1}{p_1} \cdots \binom{l_{s-1}}{p_{s-1}} \binom{l_{s+1}}{p_{s+1}} \cdots \binom{l_n}{p_n} \delta \frac{(m^{(i,j)} + n^{(i,j)})_{\substack{p_1 \dots p_{s-1} \, p_{s+1} \dots p_n \\ u_1 \dots u_{s-1} \, u_{s+1} \dots u_n}}}{m^{(i,j)} + n^{(i,j)}} \\
&\qquad \cdot F^{(i,j)}_{\substack{l_1-p_1 \dots l_s \dots l_n-p_n \\ u_1 \dots u_s \dots u_n}} \\
&\text{(c)} \quad - \sum_{\substack{p_1=1,\dots,l_1 \\ \dots\dots\dots \\ p_n=1,\dots,l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} \delta \frac{(m^{(i,j)} + n^{(i,j)})_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{m^{(i,j)} + n^{(i,j)}} \cdot F^{(i,j)}_{\substack{l_1-p_1 \dots l_n-p_n \\ u_1 \dots u_n}} \\
&\text{(d)} \quad - \sum_{s=1}^{n} \sum_{r \neq s}^{p_r=1,\dots,l_r} \binom{l_1}{p_1} \cdots \binom{l_{s-1}}{p_{s-1}} \binom{l_{s+1}}{p_{s+1}} \cdots \binom{l_n}{p_n} \frac{\overline{(m^{(i,j)} + n^{(i,j)})}_{\substack{p_1 \dots p_{s-1} \, p_{s+1} \dots p_n \\ u_1 \dots u_{s-1} \, u_{s+1} \dots u_n}}}{\overline{m^{(i,j)} + n^{(i,j)}}} \\
&\qquad (\delta F^{(i,j)}_{\substack{l_1-p_1 \dots l_s \dots l_n-p_n \\ u_1 \dots u_s \dots u_n}} + D^{(i,j)}_{\substack{l_1-p_1 \dots l_s \dots l_n-p_n \\ u_1 \dots u_s \dots u_n}}) \\
&\text{(e)} \quad - \sum_{\substack{p_1=1,\dots,l_1 \\ \dots\dots\dots \\ p_n=1,\dots,l_n}} \binom{l_1}{p_1} \cdots \binom{l_n}{p_n} \frac{\overline{(m^{(i,j)} + n^{(i,j)})}_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{\overline{m^{(i,j)} + n^{(i,j)}}} (\delta F^{(i,j)}_{\substack{l_1-p_1 \dots l_n-p_n \\ u_1 \dots u_n}} + D^{(i,j)}_{\substack{l_1-p_1 \dots l_n-p_n \\ u_1 \dots u_n}}).
\end{aligned}
\right\} \quad (10)
$$

It is evident that (a) will vanish if

$$
\delta \frac{m^{(i,j)}_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{m^{(i,j)} + n^{(i,j)}} = \delta \frac{n^{(i,j)}_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{m^{(i,j)} + n^{(i,j)}} = 0 \quad (p_1=0,\dots,l_1;\dots;\ p_n=0,\dots,l_n). \tag{11}
$$

The conditions (11) imply

$$
\delta \frac{(m^{(i,j)} + n^{(i,j)})_{\substack{p_1 \dots p_n \\ u_1 \dots u_n}}}{m^{(i,j)} + n^{(i,j)}} = 0 \quad (p_1=0,\dots,l_1;\dots;\ p_n=0,\dots,l_n),
$$

so that (b) and (c) vanish as a result. Moreover, (11) also imply

$$
\delta F^{(i,j)}_{\substack{l_1-p_1 \dots l_s \dots l_n-p_n \\ u_1 \dots u_s \dots u_n}} + D^{(i,j)}_{\substack{l_1-p_1 \dots l_s \dots l_n-p_n \\ u_1 \dots u_s \dots u_n}} = 0,
$$

$(p_1=0,\dots,l_1;\dots;\ p_n=0,\dots,l_n$; except that not all $p's=0$; $s=1,\dots,n)$, by virtue of (10) for successive values of the *l's*. Hence (d) and (e) also vanish as a result of (11). Therefore, *a sufficient condition that* $F^{(i,j)}$ *may*

*replace* $[D^{(i,j)}]$ *or* $\{D^{(i,j)}\}$ *in any seminvariant without affecting its seminvariance, is that for each pair* $(i, j=1, \ldots, n;\ i<j)$,

$$\frac{m^{(i,j)}_{u_1 \ldots u_n}}{m^{(i,j)}+n^{(i,j)}} \text{ and } \frac{n^{(i,j)}_{u_1 \ldots u_n}}{m^{(i,j)}+n^{(i,j)}} \left(m^{(i,j)}+n^{(i,j)} \neq 0;\ p_1=0, \ldots, l_1; \ldots; p_n=0, \ldots, l_n\right)$$

*be seminvariants, where* $l_r$ *is the highest order of a derivative of* $[D^{(i,j)}]$ *(or* $\{D^{(i,j)}\}$*) with respect to* $u_r$ $(r=1, \ldots, n)$ *occurring in the original seminvariant.*

### 6. *Application of the New Method to Special Cases.*

It should be noted that while we have indicated a large class of expressions which may be used in the place of $[D^{(i,j)}]$ or $\{D^{(i,j)}\}$, if we wish the simplest set of seminvariants we will set $m^{(i,j)}=1$, $n^{(i,j)}=0$, or $m^{(i,j)}=0$, $n^{(i,j)}=1$.

Let us now illustrate this method of computing seminvariants by the case of plane nets. We assume temporarily $a'_u=b'_v$, and take $\lambda_u/\lambda=b'$, $\lambda_v/\lambda=a'$. Substitution of these values in (3) yields the fundamental set of seminvariants

$$\left.\begin{array}{lll} A_1=a-2b', & B_1=b, & C_1=c+ab'+a'b-b'_u-b'^2,^* \\ A'_1=0, & B'_1=0, & C'_1=c'+a'b'-b'_v, \text{ or } c'+a'b'-a'_u, \\ A''_1=a'', & B''_1=b''-2a', & C''_1=c''+a''b'+a'b''-a'_v-a'^2,^* \end{array}\right\} \quad (12)$$

which are considerably simpler than those of (4). A further advantage of the new process over the old becomes evident when we discuss the invariants of (1). Of the seminvariants (4), only two, $A''$ and $B$, are invariants. Of the seminvariants (12), *five* are invariants, viz., those just mentioned and in addition $C_1=\mathfrak{C}$, $C'_1=K$ (or $H$), $C''_1=\mathfrak{C}''$. That this advantage can be guaranteed in general, however, is not evident. The simplicity of the new process might perhaps give it a better chance than the old to yield a larger number of seminvariants which are invariants as well. Another pseudo-canonical form, viz., that determined by $\lambda_u/\lambda=\frac{1}{2}a$, $\lambda_v/\lambda=\frac{1}{2}b''$, yields a fundamental set of seminvariants which are simpler than those of (4), but not quite so simple as those of (12). Only two of these are invariants.

In the case of conjugate systems of curves on a curved surface,† the new process (for a certain choice of $\lambda$) gives seminvariants which are very much simpler than those given by the regulation method. Moreover, three of the

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* It happens that $C_1$ and $C_1''$ may be still further simplified by use of the integrability conditions.

† G. M. Green, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVII (1915), pp. 215–246.

five are seminvariants, as against one out of five for the old method. In the case of developable surfaces,* besides yielding simpler seminvariants, the new method gives two invariants, while the other gives none.

It seems evident that the method we have outlined will be of even greater value in the cases of higher dimensional configurations. For example, the simplification due to the new process for the case of curvilinear coordinates in $n$ dimensions † is seen by noticing that it would replace, for every $i$,

$$f^{(i)} = \sum_{j=1}^{n} a_j^{(j,\,i)}$$

by some single coefficient $a_j^{(j,\,i)}$.

ANN ARBOR, MICH., *January* 2, 1918.

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* W. W. Denton, *Trans. Amer. Math, Soc.*, Vol. XIV (1913), pp. 175–208.

† G. M. Green, "The Linear Dependence of Functions," etc. *loc. cit.* Cf. especially Section 7, and equations (21).

# *On a Method for Determining the Non-Stationary State of Heat in an Ellipsoid.*

By Bibhutibhusan Datta.

## *Introduction.*

1. The first writer, who attempted, with some success, the problem of the determination of the non-stationary state of heat in an *ellipsoid with three unequal axes*, was E. Mathieu,* who showed how the problem could be reduced to the solution of certain ordinary linear differential equations. But he found these equations to be so unmanageable that he contented himself with approximating to their solutions for the special case of an *ellipsoid of revolution*. Prof. C. Niven improved upon the results of Mathieu in certain respects in an interesting memoir,† entitled "On the Conduction of Heat in Ellipsoids of Revolution."

In the present paper, I propose (1) to obtain the chief results of Prof. Niven by an entirely different method, and (2), to show how this method can be applied to the case of the ellipsoid with three unequal axes, to obtain similar results which are believed to be new. It may be noted here that, in Art. 6, I point out a mistake in Prof. Niven's memoir.

## *Preliminary Remarks and Definitions.*

2. Let the initial temperature of the ellipsoid be $f(x, y, z)$ and let its boundary, viz., $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, be maintained at temperature zero. Then the required temperature $V(x, y, z, t)$ is such that

$$\frac{\partial V}{\partial t}=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}, \tag{1}$$

* *Cours de physique mathématique*, Ch. IX.

† *Phil. Trans.*, Vol. CLXXI (1880).

the units being so chosen that the diffusivity is 1,

$$V=0 \qquad \text{on the boundary,} \tag{2}$$

$$V=f(x, y, z) \text{ when } t=0. \tag{3}$$

Thus $V$ can be expressed as a sum of terms of the form $Ae^{-\lambda^2 t}W(x, y, z)$, where the normal function $W$ satisfies the equation

$$\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial^2 W}{\partial z^2}=-\lambda^2 W,$$

and vanishes on the boundary; and the constants $A$ are so chosen that the initial condition (3) is satisfied, so that $f(x, y, z)=\Sigma AW$.

When $a=b=c$ so that the ellipsoid becomes a sphere of radius $a$, an appropriate normal function is

$$S_n(\lambda r)P_n^m(\cos\theta)\cos m\phi,$$

where

$$S_n(x)=(-1)^n\sqrt{\frac{\pi}{2}}x^{-\frac{1}{2}}J_{n+\frac{1}{2}}(x), \quad P_n^m(\cos\theta)=\sin^m\theta\frac{d^mP_n}{d\mu^m}, \ \mu \text{ being } \cos\theta,$$

and $\lambda$ is a root of the equation $S_n(\lambda a)=0$. Throughout the present paper I will represent by $W_n^m$ the normal function corresponding to the function $S_n(\lambda r)P_n^m(\cos\theta)\cos m\phi$, and denote $W_n^0$ by $W_n$.

I proceed now to obtain the functions $W$ of various types.

### *$W_0$ For Ellipsoid of Revolution.*

3. Let $e$ denote the eccentricity of the ellipsoid, then neglecting $e^4$ and higher powers, the equation of the ellipsoid can be written as

$$r=a\{1-\tfrac{1}{3}e^2+\tfrac{1}{3}e^2P_2(\cos\theta)\}, \quad i.\,e., \quad r=\alpha(1+\varepsilon P_2(\cos\theta)),$$

where $\alpha=a(1-\frac{1}{3}e^2)$ and $\varepsilon=\frac{1}{3}e^2$. Now assume that

$$W_0=S_0(\lambda r)+\varepsilon\sum_{t=1}^{\infty}M_tS_t(\lambda r)P_t(\cos\theta),$$

$M_t$ being an unknown constant to be determined. Then, evidently, $W_0$ satisfies the partial differential equation (I); and to satisfy the boundary condition we must have

$$0=S_0(\lambda\alpha)+\varepsilon\alpha\frac{\partial S_0(\lambda\alpha)}{\partial\alpha}P_2+\varepsilon\sum_{t=1}^{\infty}M_tS_t(\lambda\alpha)P_t(\cos\theta),$$

since $\varepsilon^2$ and higher powers are neglected. This equation must hold for all

values of cos $\theta$. Therefore, equating to zero the coefficients of the various zonal harmonics, we get

$$S_0(\lambda a) = 0, \tag{1}$$

$$M_2 S_2(\lambda a) + a \frac{\partial S_0(\lambda a)}{\partial a} = 0, \tag{2}$$

and all the other $M$'s are zero. Hence the required expression for $W_0$, in terms of $a$ and $e$, is

$$W_0 = S_0(\lambda r) - \tfrac{1}{3} e^2 \frac{a \dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)} S_2(\lambda r) P_2(\cos\theta),$$

where $\lambda$ is given by the equation (1). But the general solution of the equation (1) is known to be

$$\lambda a = i\pi,$$

$i$ being any integer. Hence

$$\lambda a = i\pi + \tfrac{1}{3} i\pi e^2.$$

4. In order to obtain a closer approximation to $W_0$, we will retain $e^4$ and neglect the sixth and higher powers. Thus the equation of the ellipsoid is

$$r = a[(1 - \tfrac{1}{3} e^2 - \tfrac{2}{15} e^4) + (\tfrac{1}{3} e^2 + \tfrac{1}{21} e^4) P_2 + \tfrac{3}{35} e^4 P_4], \text{ i. e., } r = \beta[1 + \sigma P_2 + \tau P_4],$$

where $\beta = a(1 - \tfrac{1}{3} e^2 - \tfrac{2}{15} e^4)$, $\sigma = \tfrac{1}{3} e^2 + \tfrac{10}{63} e^4$, and $\tau = \tfrac{3}{35} e^4$.

Let us assume that

$$W_0 = S_0(\lambda r) - \tfrac{1}{3} e^2 \frac{a \dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)} S_2(\lambda r) P_2(\cos\theta) + \tau \sum_{i=1}^{\infty} N_i S_i(\lambda r) P_i(\cos\theta),$$

$N_i$ being an unknown constant to be determined. Expanding by Taylor's Theorem, we have

$$S_0(\lambda r) = S_0(\lambda\beta) + \beta \frac{\partial S_0(\lambda\beta)}{\partial \beta} (\sigma P_2 + \tau P_4) + \tfrac{1}{2!} \beta^2 \frac{\partial^2 S_0(\lambda\beta)}{\partial \beta^2} (\sigma P_2 + \tau P_4)^2 + \ldots,$$

and

$$S_2(\lambda r) = S_2(\lambda\beta) + \beta \frac{\partial S_2(\lambda\beta)}{\partial \beta} (\sigma P_2 + \tau P_4) + \ldots,$$

when

$$r = \beta(1 + \sigma P_2 + \tau P_4).$$

Again

$$(P_2)^2 = \tfrac{18}{35} P_4 + \tfrac{2}{7} P_2 + \tfrac{1}{5}.$$

Hence we must have

$$0=S_0(\lambda\beta)+\beta\frac{\partial S_0(\lambda\beta)}{\partial\beta}(\sigma P_2+\tau P_4)+\frac{\sigma^2}{2}\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}\left[\tfrac{18}{35}P_4+\tfrac{2}{7}P_2+\tfrac{1}{5}\right]$$
$$-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda\beta)P_2-\tfrac{1}{3}e^2\sigma\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}\left[\tfrac{18}{35}P_4+\tfrac{2}{7}P_2+\tfrac{1}{5}\right]$$
$$+\tau\sum_{i=1}^{\infty}N_iS_i(\lambda\beta)P_i.$$

This equation must hold for all values of $\cos\theta$. Therefore, equating to zero the coefficients of the various zonal harmonics, we get

$$S_0(\lambda\beta)+\tfrac{1}{10}\sigma^2\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}-\tfrac{1}{15}e^2\sigma\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}=0, \tag{1}$$

$$\tau N_2S_2(\lambda\beta)+\sigma\beta\frac{\partial S_0(\lambda\beta)}{\partial\beta}+\tfrac{1}{7}\sigma^2\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot S_2(\lambda\beta)$$
$$-\tfrac{2}{21}e^2\sigma\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}=0, \tag{2}$$

$$\tau N_4S_4(\lambda\beta)+\tau\beta\frac{\partial S_0(\lambda\beta)}{\partial\beta}+\tfrac{9}{35}\sigma^2\beta^2\frac{\partial^2 S_0(\lambda\beta)}{\partial\beta^2}$$
$$-\tfrac{6}{35}e^2\sigma\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot\beta\frac{\partial S_2(\lambda\beta)}{\partial\beta}=0; \tag{3}$$

and all the other $N$'s are zero. On substituting the values of $\beta$, $\sigma$ and $\tau$ in terms of $a$ and $e$, the above equations take the forms

$$S_0(\lambda a)-(\tfrac{1}{3}e^2+\tfrac{2}{15}e^4)a\frac{\partial S_0(\lambda a)}{\partial a}+\tfrac{1}{15}e^4a^2\frac{\partial^2 S_0(\lambda a)}{\partial a^2}$$
$$-\tfrac{1}{45}e^4\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\,a\frac{\partial S_2(\lambda a)}{\partial a}=0, \tag{4}$$

$$\tfrac{3}{35}N_2S_2(\lambda a)+\tfrac{1}{21}a\frac{\partial S_0(\lambda a)}{\partial a}-\tfrac{2}{21}a^2\frac{\partial^2 S_0(\lambda a)}{\partial a^2}+\tfrac{5}{63}\cdot\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot a\frac{\partial S_2(\lambda a)}{\partial a}=0, \tag{5}$$

$$\tfrac{3}{35}N_4S_4(\lambda a)+\tfrac{3}{35}a\frac{\partial S_0(\lambda a)}{\partial a}+\tfrac{1}{35}a^2\frac{\partial^2 S_0(\lambda a)}{\partial a^2}-\tfrac{2}{35}\cdot\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}\cdot a\frac{\partial S_2(\lambda a)}{\partial a}=0. \tag{6}$$

Hence the required expression for $W_0$ is

$$W_0=S_0(\lambda r)-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda r)P_2(\cos\theta)+\tfrac{3}{35}e^4N_2S_2(\lambda r)P_2(\cos\theta)$$
$$+\tfrac{3}{35}e^4N_4S_4(\lambda r)P_4(\cos\theta),$$

the values of $N_2$ and $N_4$ being given by the equations (5) and (6), and the value of $\lambda$ being determined from the equation (4).

5. I will now proceed with the solution of the equation (4) of the preceding article. If we neglect $e$ altogether, the equation reduces to $S_0(\lambda a)=0$, whose roots are given by $\lambda a=i\pi$. Therefore let the full value of $\lambda a$ be

$$i\pi+l_1e^2+l_2e^4+\ldots,$$

$l_1$, $l_2$ being unknown quantities which are to be determined. $S_n$ can be expanded in a series containing a finite number of terms so that

$$S_n(x)=\left[\frac{1}{x}-\frac{n'(n'-1')}{1.2}\cdot\frac{1}{2^2x^3}\right.$$
$$\left.+\frac{n'(n'-1')(n'-2')(n'-3')}{1.2.3.4}\cdot\frac{1}{2^4x^5}-\ldots.\right]\sin\left(x+\frac{n\pi}{2}\right)$$
$$+\left[\frac{n'}{1}\cdot\frac{1}{2x^2}-\frac{n'(n'-1')(n'-2')}{1.2.3}\cdot\frac{1}{2^3x^4}+\ldots.\right]\cos\left(x+\frac{n\pi}{2}\right),$$

where $n'$ stands for $n(n+1)$ and $(n'-t')$ for $(n-t)(n+t+1)$. Hence we have the following:

$$S_0(\lambda a)=\frac{l_1e^2}{i\pi}\cos i\pi+\left(\frac{l_2}{i\pi}-\frac{l_1^2}{i^2\pi^2}\right)\cos i\pi\cdot e^4+\ldots.,$$

$$S_2(\lambda a)=-\frac{3}{i^2\pi^2}\cos i\pi+\ldots., \quad a\frac{\partial S_0(\lambda a)}{\partial a}=\cos i\pi-\frac{l_1}{i\pi}\cos i\pi\cdot e^2+\ldots.,$$

$$a^2\frac{\partial^2 S_0(\lambda a)}{\partial a^2}=-2\cos i\pi+\ldots., \quad a\frac{\partial S_2(\lambda a)}{\partial a}=-\left(1-\frac{9}{i^2\pi^2}\right)\cos i\pi+\ldots.,$$

$$a\frac{\partial S_0(\lambda a)}{\partial a}\cdot a\frac{\partial S_2(\lambda a)}{\partial a}\Big/S_2(\lambda a)=\tfrac{1}{3}(i^2\pi^2-9)\cos i\pi+\ldots..$$

Therefore, substituting the above expressions for $S_0(\lambda a)$, etc., in the equation (4) of the preceding article, and equating to zero the coefficients of $e^2$ and $e^4$, we obtain finally

$$\lambda a=i\pi+\tfrac{1}{3}i\pi e^2+\frac{i\pi}{135}(i^2\pi^2+27)e^4+\ldots..$$

*Comparison with the Results of Niven.*

6. By neglecting $e^6$ and higher powers, Prof. C. Niven has obtained an expression * for the parameter $\lambda$, viz.:

$$\lambda a = i\pi + \tfrac{1}{3} i\pi e^2 + \frac{i\pi}{405}(i^2\pi^2+27)e^4 + \ldots,$$

which differs from that obtained by me in the preceding article only in so far as the coefficient of $e^4$ is concerned. After carefully going through Prof. Niven's calculations, I find that the mistake of Prof. Niven must be attributed to some inadvertence on his part, as, by repeating his process, I get the correct result.

From the identity of the values of $\lambda$, it follows at once that my expression for $W_0$ and Prof. Niven's must be identical. For, as is well-known, for the same value of $\lambda$ there can not be two different solutions of the equation

$$\frac{\partial^2 W_0}{\partial x^2} + \frac{\partial^2 W_0}{\partial y^2} + \frac{\partial^2 W_0}{\partial z^2} + \lambda^2 W_0 = 0,$$

both of which vanish on the boundary.

*$W_n$ For Ellipsoid of Revolution, $n>0$.*

7. Let $e^4$ and higher powers be neglected, so that the equation of the ellipsoid is the same as in Art. 3, viz.: $r=a\{1+\varepsilon P_2(\cos\theta)\}$. Then assume that

$$W_n = S_n(\lambda r)P_n(\cos\theta) + \varepsilon \sum_{t=0}^{\infty}{}' H_t S_t(\lambda r) P_t(\cos\theta),$$

where the $H$'s are unknown constants, to be determined, and $\sum_{t=0}^{\infty}{}'$ refers to all the values of $t$ except $t=n$. Thus it is evident that $W_n$ satisfies the partial differential equation (I), and it remains to find the values of the constants $H$'s so as to satisfy the boundary condition. Since $\varepsilon^2$ and higher powers of $\varepsilon$ are to be neglected, we have, on putting $r=a(1+\varepsilon P_2)$,

$$S_n(\lambda r) = S_n(\lambda a) + \varepsilon a \frac{\partial S_n(\lambda a)}{\partial a} P_2;$$

also, we have

$$P_2 \cdot P_n = B_{n+2}P_{n+2} + C_n P_n + D_{n-2}P_{n-2},$$

where

$$B_{n+2} = \tfrac{3}{2} \cdot \frac{(n+1)(n+2)}{(2n+3)(2n+1)}, \quad C_n = \frac{n(n+1)}{(2n+3)(2n-1)},$$

$$D_{n-2} = \tfrac{3}{2} \cdot \frac{n(n-1)}{(2n+1)(2n-1)}.$$

* *Loc. cit.*, p. 145.

Hence we must have

$$0=S_n(\lambda\alpha)P_n+\varepsilon\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}[B_{n+2}P_{n+2}+C_nP_n+D_{n-2}P_{n-2}]+\varepsilon\sum_{t=0}^{\infty}{}'H_tS_t(\lambda\alpha)P_t.$$

This must be true for all values of $\cos\theta$. Therefore equating to zero the coefficients of the various zonal harmonics, we get

$$S_n(\lambda\alpha)+\varepsilon\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}C_n\ =0, \tag{1}$$

$$H_{n+2}S_{n+2}(\lambda\alpha)+\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}B_{n+2}=0, \tag{2}$$

$$H_{n-2}S_{n-2}(\lambda\alpha)+\alpha\frac{\partial S_n(\lambda\alpha)}{\partial\alpha}D_{n-2}=0, \tag{3}$$

and all the other $H$'s are zero. Thus the unknown constants are determined and the required expression for $W_n$, in terms of $a$ and $e$, is

$$W_n=S_n(\lambda r)P_n-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_n(\lambda a)}{\partial a}}{S_{n+2}(\lambda a)}S_{n+2}(\lambda r)P_{n+2}B_{n+2}$$
$$-\tfrac{1}{3}e^2\frac{a\dfrac{\partial S_n(\lambda a)}{\partial a}}{S_{n-2}(\lambda a)}S_{n-2}(\lambda r)P_{n-2}D_{n-2};$$

$\lambda$ being a root of the equation (1).

But, in terms of $a$ and $e$, this equation is

$$S_n(\lambda a)-\frac{n^2+n-1}{(2n+3)(2n-1)}e^2a\frac{\partial S_n(\lambda a)}{\partial a}=0. \tag{4}$$

Therefore, if $\varkappa$ be a root of the equation $S_n(x)=0$, the corresponding solution of (4) is given by

$$\lambda a=\varkappa\left\{1+\frac{n^2+n-1}{(2n+3)(2n-1)}e^2\right\}.$$

*$W_n^m$ For Ellipsoid of Revolution.*

8. Let $e^4$ and higher powers be neglected, and assume that

$$W_n^m=S_n(\lambda r)P_n^m(\cos\theta)\cos m\phi+\varepsilon\sum_{t=m}^{\infty}{}'I_t^mS_t(\lambda r)P_t^m(\cos\theta)\cos m\phi,$$

where $\sum_{t=m}^{\infty}{}'$ refers to all values of $t$ from $m$ up to infinity except the value $t=n$, and $I_t^m$ is an unknown constant to be determined. Then it is evident that $W_n^m$

satisfies the partial differential equation (I). Now, putting $r=a(1+\varepsilon P_2)$, we get

$$S_n(\lambda r)=S_n(\lambda a)+\varepsilon a\frac{\partial S_n(\lambda a)}{\partial a}P_2;\ \text{and}\ P_2\cdot P_n^m=B_{n+2}^m P_{n+2}^m+C_n^m P_n^m+D_{n-2}^m P_{n-2}^m,$$

where

$$B_{n+2}^m=\tfrac{3}{2}\cdot\frac{(n-m+1)(n-m+2)}{(2n+3)(2n+1)},\quad C_n^m=\frac{n(n+1)-3m^2}{(2n+3)(2n-1)},$$

$$D_{n-2}^m=\tfrac{3}{2}\cdot\frac{(n+m)(n+m-1)}{(2n+1)(2n-1)}.$$

Hence, from the boundary condition, we must have

$$0=S_n(\lambda a)P_n^m\cos m\phi+\varepsilon a\frac{\partial S_n(\lambda a)}{\partial a}[B_{n+2}^m P_{n+2}^m+C_n^m P_n^m+D_{n-2}^m P_{n-2}^m]\times\cos m\phi+\varepsilon\sum_{t=m}^{\infty}{}'I_t^m S_t(\lambda a)P_t^m\cos m\phi.$$

Therefore, equating to zero the coefficients of the various surface harmonics, we get

$$S_n(\lambda a)+\varepsilon a\frac{\partial S_n(\lambda a)}{\partial a}C_n^m=0, \tag{1}$$

$$I_{n+2}^m S_{n+2}(\lambda a)+a\frac{\partial S_n(\lambda a)}{\partial a}B_{n+2}^m=0, \tag{2}$$

$$I_{n-2}^m S_{n-2}(\lambda a)+a\frac{\partial S_n(\lambda a)}{\partial a}D_{n-2}^m=0, \tag{3}$$

and all the other $I_t^m$'s are zero.

Thus the required expression for $W_n^m$ in terms of $a$ and $e$ is

$$W_n^m=\left[S_n(\lambda r)P_n^m(\cos\theta)-\tfrac{1}{3}e^2\frac{a\frac{\partial S_n(\lambda a)}{\partial a}}{S_{n+2}(\lambda a)}S_{n+2}(\lambda r)P_{n+2}^m(\cos\theta)B_{n+2}^m-\tfrac{1}{3}e^2\frac{a\frac{\partial S_n(\lambda a)}{\partial a}}{S_{n-2}(\lambda a)}S_{n-2}(\lambda r)P_{n-2}^m(\cos\theta)D_{n-2}^m\right]\cos m\phi,$$

where $\lambda$ is a root of the equation (1). But, expressed in terms of $a$ and $e$, this equation becomes

$$S_n(\lambda a)-\frac{(n^2+n-1)+m^2}{(2n+3)(2n-1)}e^2a\frac{\partial S_n(\lambda a)}{\partial a}=0. \tag{4}$$

Therefore, we obtain

$$\lambda a=x\left\{1+\frac{(n^2+n-1)+m^2}{(2n+3)(2n-1)}e^2\right\},$$

corresponding to the root $x$ of $S_n(x)=0$.

*$W_0$ For Ellipsoid with Three Unequal Axes.*

9. Let $e_1$ and $e_2$ be the eccentricities of the two principal diametral sections of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $a>b>c$, by planes passing through the major axis. Then, neglecting the fourth and higher powers of $e_1$ and $e_2$, the equation to the ellipsoid is

$$r=a[1-\tfrac{1}{6}(e_1^2+e_2^2)(P_0-P_2)+\tfrac{1}{12}(e_2^2-e_1^2)P_2^2\cos 2\phi],$$

*i. e.*,
$$r=\gamma(1+\varepsilon_1 P_2+\varepsilon_2 P_2^2\cos 2\phi),$$

where $\gamma=a\{1-\frac{1}{6}(e_1^2+e_2^2)\}$, $\varepsilon_1=\frac{1}{6}(e_1^2+e_2^2)$, and $\varepsilon_2=\frac{1}{12}(e_2^2-e_1^2)$.

Now assume that

$$W_0=S_0(\lambda r)+\sum_{t=m}^{\infty}{}'\sum_{m=0}^{\infty}R_{m,t}S_t(\lambda r)P_t^m(\cos\theta)\cos m\phi,$$

where $R_{m,t}$ is an unknown constant to be determined and $\sum_{t=m}^{\infty}{}'$ refers to all values of $t$ from $m$ up to $\infty$, except $t=0$. Then it is evident that $W_0$ satisfies the partial differential equation (I). To satisfy the boundary condition, we must have

$$0=S_0(\lambda\gamma)+\gamma\frac{\partial S_0(\lambda\gamma)}{\partial\gamma}(\varepsilon_1 P_2+\varepsilon_2 P_2^2\cos 2\phi)+\sum_{t=m}^{\infty}{}'\sum_{m=0}^{\infty}R_{m,t}S_t(\lambda\gamma)P_t^m(\cos\theta)\cos m\phi,$$

the unknown constants $R's$ being assumed to be of the same order as $\varepsilon_1$ and $\varepsilon_2$. Therefore, equating to zero the coefficients of the various surface harmonics, we get

$$S_0(\lambda\gamma)=0, \tag{1}$$

$$R_{0,2}S_2(\lambda\gamma)+\varepsilon_1\gamma\frac{\partial S_0(\lambda\gamma)}{\partial\gamma}=0, \tag{2}$$

$$R_{2,2}S_2(\lambda\gamma)+\varepsilon_2\gamma\frac{\partial S_0(\lambda\gamma)}{\partial\gamma}=0; \tag{3}$$

and all the other $R's$ are zero. Thus the $R's$ are determined, and we get finally the required expression for $W_0$ to be

$$W_0=S_0(\lambda r)-\tfrac{1}{6}(e_1^2+e_2^2)\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda r)P_2(\cos\theta)$$

$$+\tfrac{1}{12}(e_1^2-e_2^2)\frac{a\dfrac{\partial S_0(\lambda a)}{\partial a}}{S_2(\lambda a)}S_2(\lambda r)P_2^2(\cos\theta)\cos 2\phi,$$

where $\lambda$ is a root of the equation (1). But the roots of the equation (1) are given by

$$\lambda\gamma = i\pi.$$

Hence, in terms of $a$ and the eccentricities, we have

$$\lambda a = i\pi + \tfrac{1}{6}(e_1^2 + e_2^2)\, i\pi.$$

*Conclusion.*

10. I conclude this paper by pointing out that the results or Arts. 3–9 admit of extension and generalization in various directions. For example, a procedure similar to that of Art. 9 will give us $W_n^m$ for the ellipsoid with three unequal axes. Also, denoting by $\overline{W}_n^m$, the normal function corresponding to

$$S_n(\lambda r)\, P_n^m(\cos\theta) \sin m\phi,$$

it is obvious that, for the ellipsoid of revolution as well as for the ellipsoid with three unequal axes, $\overline{W}_n^m$ can be obtained in the same way as $W_n^m$.

UNIVERSITY OF CALCUTTA, 1917.

# *Nilpotent Algebras* Generated by Two Units, i and j, Such That $i^2$ Is Not an Independent Unit.*

By Guy Watson Smith.

## I. *Introduction.*

The problem of referring all hypercomplex number systems to a relatively small number of typical forms was suggested by Hamilton,† but with the exception of De Morgan's discussion of double and triple algebras, nothing much was done till Benjamin Peirce‡ worked out all algebras of deficiency zero and one. Starkweather§ worked out algebras of deficiency two. He showed also that algebras of $n$ units could be obtained from those of $n-1$ units. Cartan‖ using the characteristic equation developed the semi-simple and the nilpotent sub-algebras, and showed the possibility of representing every algebra by means of units with double character. Taber¶ reestablished the results of Peirce and extended them to any domain of rationality for the coordinates. Wedderburn** and Voghera†† made an advance in the treatment of the hypercomplex algebra by basing their work on the conception of invariant classes of numbers in the algebra.

Besides this direct line of development there have been two others. The first is by means of the continuous group, the second is by using the matrix

* This paper considers only linear associative algebras whose coordinates are taken from the field of ordinary complex numbers.

† "Lectures on Quaternions," Preface, pp, 29–31.

‡ "Linear Associative Algebra," American Journal of Mathematics, Vol. IV (1881) pp. 97–192.

§ "Non-Quaternion Number Systems Containing no Skew Units," American Journal of Mathematics, Vol. XXI (1899), pp. 369–386.

‖ "Les groupes bilineaires et les systemes de nombres complexes, *Annales de la Facultedes Sciences de Toulouse*, Vol. XII (1898), B. pp. 1–99.

¶ "On Hypercomplex Number Systems," *Am. Math. Soc. Trans.*, Vol. V (1904), pp. 509–548.

** "On Hypercomplex Numbers," *London Math. Soc. Proceedings*, Series 2, Vol. VI (1907), pp. 77–118.

†† "Zusammenstellung der Irreduziblen Komplexen Zahlensysteme in sechs Einheiten," *Denkschriften der Math. Nat. Klasse der Kais. Akad. der Wiss.* Wien., Vol. LXXXIV (1908), pp. 269–329.

theory. The first method was used by Scheffers,* Molien,† and Study,‡ the second by Shaw § and Frobenius. ||

Shaw regards all associative numbers as belonging to an associative algebra of an infinite number of units, the "associative units," $\lambda_{rst}$, which are elementary matrices. Each associative number is a linear combination of these units, so the theory of linear associative algebra is the theory of these associative units. He shows that the presence of a modulus is not necessary, thus making the methods particularly applicable to nilpotent systems. He proves that the equation of an algebra determines all the units but those which form a nilpotent system, and consequently to get all linear associative algebras we must first determine all nilpotent algebras. Benjamin Peirce ¶ was the first to recognize the importance of nilpotent algebras. Furthermore, algebras of order $n$ may be found without first knowing those of order $n-1$. By selecting a base and adjoining to this a nilpotent unit, an ever-increasing system of nilpotent algebras may be determined.

The simplest case of the Shaw canonical form ** of a nilpotent algebra is that in which there is only a single generator, $j$, whose $\mu_1$-th power vanishes,

$$j, j^2, j^3, j^4, \ldots, j^{\mu_1-1}.$$

Three lemmas concerning polynomials in this nilpotent have been introduced for use in the handling of the next simplest case, namely, that in which there are two generators, $j$ and another nilpotent unit, $i$, whose square is not an independent unit in the canonical form. The unit $j$ is such that $j^{\mu_1-1} \neq 0$, $j^{\mu_1}=0$ and $ij^{\mu_2-1} \neq 0$, $ij^{\mu_2}=0$, where $\mu_1$ and $\mu_2$ are multiplicities of $j$ relative to $j$ and to $i$ respectively, and $\mu_1 \geqq \mu_2$. The expressions $i, j^s, ij^t$ where $0<s<\mu_1$, and $0<t<\mu_2$, are the independent units of the system. The algebra is therefore of order $\mu_1+\mu_2-1$, or if we insert a modulus $\eta$, of order $\mu_1+\mu_2$. The $\mu_1$ power of every number must vanish, hence the deficiency in this case is $\mu_2$. There is at least one hypernumber which does not vanish for a power lower than $\mu_1$,

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* "Zurückführung complexer Zahlensysteme auf typische Formen," *Math. Annalen,* Vol. XXXIX (1891), pp. 293–390.

† "Ueber Systeme höherer complexen Zahlen," *Math. Annalen,* Vol. XLI (1893) pp. 83–156.

‡ "Ueber Systeme von complexen Zahlen," *Gött. Nach.* (1889), pp. 237–268. "Complexe Zahlen und Transformationsgruppen," *Leipzig Berichte,* Vol. XLI (1889), pp. 177–228.

§ "Theory of Linear Associative Algebra," *Am. Math. Soc. Trans.,* Vol. IV (1903) pp. 251–287. "On Nilpotent Algebras," *Am. Math. Soc. Trans.,* Vol. IV (1903), pp. 405–422.

|| "Theorie der hypercomplexen Grossen," *Berlin Berichte* (1903), pp. 504–537, 634–645.

¶ *Loc. cit.,* p. 118.

** *Loc. cit.,* p. 406.

and there may be several such. The base unit $i$ is chosen such that $ij^{\mu_2}=0$, $ij^{\mu_2-1}\neq 0$, the various products $ij, ij^2, \ldots$, do not contain terms in $j$ alone, such as $aj^m$, and $\mu_2$ is such that there is no hypernumber $i'$ satisfying the conditions on $i$ for which $i'j^{\mu_2}=0$, $i'j^{\mu_2-1}\neq 0$, $\mu_3>\mu_2$. In other words, $i$ is not the product of any hypernumber into $j$, and we do not have for any power of $j$, $ij^t=bj^t$. These characters of $i$ and $j$ are essential. The two sequences $j, j^2, \ldots, j^{\mu_1-1}$ and $i, ij, ij^2, \ldots, ij^{\mu_2-1}$ constitute the two shear regions of $j$. In the cases Peirce considered, that of deficiency zero contains no $i$; that of deficiency unity must have $\mu_2=1$, hence $ij=0$. In Starkweather's types where only two generators enter, $\mu_2=2$, hence $ij$ is a unit but $ij^2=0$. In the types herein considered the deficiency does not play any rôle at all. The investigation is along the line of Shaw's construction by *generators* and not by classification by numerical invariants, other than those entering the equations of condition. The associative units are used, though not indispensable, for convenience in making reductions.

## II. *Lemmas Concerning Polynomials of a Single Nilpotent j.*

LEMMA I. *If $j$ is a nilpotent number ($j^{\mu_1}=0$), and if $F(j)$ and $G(j)$ are polynomials in $j$, a quotient $Q(j)$ can always be obtained $\frac{F(j)}{G(j)}$ provided $F(j)$ does not contain a term of lower order than does $G(j)$.*

PROOF: The lemma will be proved if we can find a polynomial, $Q(j)$ of not more than $\mu_1$ terms such that $F(j)=Q(j)\cdot G(j)$, or

$$f_0+f_1j+f_2j^2+\ldots.=(q_0+q_1j+q_2j^2+\ldots.)(g_0+g_1j+g_2j^2+\ldots.).$$

Equating coefficients of corresponding powers of $j$ on both sides since they belong to independent units,

$$f_0=q_0g_0,\quad f_1=q_0g_1+q_1g_0,\quad f_2=q_0g_2+q_1g_1+q_2g_0,\quad \ldots\ldots\ldots\ldots$$

These equations can not be solved if $F(j)$ starts with a lower power of $j$ than $G(j)$, that is, if

$$f_0=f_1=f_2=\ldots.=f_h=0,\ f_{h+1}\neq 0,$$

while $g_0=g_1=\ldots.=g_{k+h}=0$ $k>0$. If, however, $f_0=f_1=\ldots.=f_h=0$, $f_{h+1}\neq 0$, while $g_0=g_1=\ldots.=g_{h-k}=0$, $g_{h-k+1}\neq 0$, $0<k<h+1$, then the first $h-k+1$ of the above equations are identically satisfied, the next $k$ equations give $q_0=q_1=\ldots.=q_{k-1}=0$ and from the remaining $\mu_1-h-1$ equations, $q_k$, $q_{k+1}, \ldots., q_{\mu_1-h+k-1}$ may be uniquely determined, leaving $q_{\mu_1-h+k}, q_{\mu_1-h+k+1}, \ldots.,$ $q_{\mu_1-1}$ arbitrary.

LEMMA II. *Consider the equation* $L(j)\cdot M(j)=0$ *where*

$$L(j)=l_0+l_1j+l_2j^2+\ldots.+l_{\mu_1-1}j^{\mu_1-1},\quad M(j)=m_0+m_1j+m_2j^2+\ldots.+m_{\mu_1-1}j^{\mu_1-1}.$$

(a). If $l_0\neq 0$, $M(j)$ vanishes entirely.

PROOF: $(l_0+l_1j+l_2j^2+\ldots.+l_{\mu_1-1}j^{\mu_1-1})(m_0+m_1j+\ldots.+m_{\mu_1-1}j^{\mu_1-1})=0.$
Then

$$l_0m_0=0\,\therefore\, m_0=0,\quad l_0m_1+l_1m_0=0\,\therefore\, m_1=0,\quad l_0m_2+l_1m_1+l_2m_2=0\,\therefore\, m_2=0,$$
$$\ldots\ldots\ldots\ldots,\quad l_0m_{\mu_1-1}+\text{vanishing terms}=0\,\therefore\, m_{\mu_1-1}=0.$$

(b). If $l_0=l_1=l_2=\ldots.=l_n=0$, $l_{n+1}\neq 0$, $n<\mu_1-1$, then $L=j^{n+1}L_{n+1}(j)$, $M(j)j^{n+1}L_{n+1}(j)=0$, where $L_{n+1}(j)=l_{n+1}+l_{n+2}j+\ldots.+l_{\mu_1-1}j^{\mu_1-n-2}$ and $l_{n+1}\neq 0$. Applying (a) to the equation $M(j)j^{n+1}\cdot L_{n+1}(j)=0$ gives

$$M(j)j^{n+1}=m_0j^{n+1}+m_1j^{n+2}+\ldots.+m_{\mu_1-n-2}j^{\mu_1-1}=0,$$
$$\therefore\ m_0=m_1=m_2=\ldots.=m_{\mu_1-n-2}=0 \text{ or } M(j)=j^{\mu_1-n-1}M_1(j),$$

where $M_1(j)=m_{10}+m_{11}j+m_{12}j^2+\ldots.+m_{1n}j^n+$ any terms whatever up to $j^{\mu_1-1}$.

LEMMA III. *If* $C(j)$ *is a polynomial in* $j$ *with non-vanishing constant term, then* $j$ *can be expressed as a polynomial in* $(jC)$.

PROOF: Let $C(j)=a_0+a_1j+a_2j^2+\ldots.+a_{\mu_1-1}j^{\mu_1-1}$, $a_0\neq 0$, then

$$jC=a_0j+a_1j^2+a_2j^3+\ldots.+a_{\mu_1-2}j^{\mu_1-1}:$$

$$(jC)^2=a_0^2j^2+2a_0a_1j^3+(2a_0a_2+a_1^2)j^4+\ldots.,\quad (jC)^3=a_0^3j^3+3a_0^2a_1j^4+\ldots.,$$
$$(jC)^4=a_0^4j^4+\ldots.,\quad \ldots\ldots\ldots\ldots\ldots\ldots$$

And since $a_0, a_0^2, a_0^3, \ldots., a_0^{\mu_1-1}\neq 0$ by hypothesis the determinant of coefficients does not vanish, and we may solve these equations for $j, j^2, j^3, \ldots., j^{\mu_1-1}$ in terms of powers of $(jC)$.

## III. *Expressions for the Generators.*

From a nilpotent system choose any nilpotent expression $j$ to be a unit, called the adjunct unit, such that $j$ is a number which has as high a non-vanishing power as any number of the system, and a set of expressions (in this paper one, say $i$), called the base. Then it is known that any hypernumber of the system is linearly expressible in terms of the $\mu_1+\mu_2-1$ independent units

$$i, j, ij, j^2, ij^2, j^3, ij^3, \ldots., ij^{\mu_2-1}, j^{\mu_2}, j^{\mu_2+1}, \ldots., j^{\mu_1-1},$$

and in this order the product of any two units $ij^a$ and $ij^b$ is linearly expressible in terms of the units which follow* the unit $ij^{a+b-1}$.

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*Shaw, *Trans. Am. Math. Soc.*, Vol. IV (1903), pp. 406–410.

It has been shown† that any number is expressible linearly in terms of elementary matrices, $\lambda_{rst}$, called "associative units,"† where

$$\lambda_{rst}\lambda_{r's't'}=c\delta_{ss'}\lambda_{rs(t+t')}, \qquad \mu_r-\mu_s\leq t<\mu_r, \quad \mu_{r'}-\mu_{s'}\leq t'<\mu_{r'},$$
$$\mu_r-1\geq t+t'\geq\mu_r-\mu_{s'}, \qquad \delta_{ss'}=1 \text{ if } s=r', \quad 0 \text{ if } s\neq r',$$

$c=1$ if $t+t'$ satisfies the above condition, 0 if $t+t'$ does not satisfy the condition, $\mu_r$ and $\mu_s$ are multiplicity numbers for $r$ and $s$ respectively, determined by the power of $j$ which equals 0, and the power of $j$ in $ij^w$ which causes it to vanish, that is here $j^{\mu_1}=0$, $j^{\mu_1-1}\neq 0$ and $ij^{\mu_2}=0$, $ij^{\mu_2-1}\neq 0$.

The frame of the nilpotent system formed upon the base $i$ and the adjunct unit $j$ is:*

$$i=\lambda_{210}+a_{221}\lambda_{221}+a_{222}\lambda_{222}+\ldots.+a_{22\mu_2-1}\lambda_{22\mu_2-1}+a_{12\mu_1-\mu_2}\lambda_{12\mu_1-\mu_2}+\ldots.+a_{12\mu_1-1}\lambda_{12\mu_1-1},$$
$$j=\lambda_{111}+b_{221}\lambda_{221}+\ldots.+b_{22\mu_2-1}\lambda_{22\mu_2-1}+b_{12\mu_1-\mu_2}\lambda_{12\mu_1-\mu_2}+\ldots.+b_{12\mu_1-1}\lambda_{12\mu_1-1}.$$

We may indicate these expressions more compactly by introducing certain symbolic polynomials which will take care of the third subscripts. Thus we write

$$\lambda_{22}\theta A\equiv a_{221}\lambda_{221}+a_{222}\lambda_{222}+\ldots.+a_{22\mu_2-1}\lambda_{22\mu_2-1},$$

where $A=a_{221}+a_{222}\theta+\ldots.+a_{22\mu_2-1}\theta^{\mu_2-2}$, and $\theta$ is a nilpotent such that $\theta^{\mu_2}=0$, $\lambda_{22i}\theta=\lambda_{22i+1}$. In a similar manner we may write

$$\lambda_{12}\delta^{\mu_1-\mu_2}B=a_{12\mu_1-\mu_2}\lambda_{12\mu_1-\mu_2}+\ldots.+a_{12\mu_1-1}\lambda_{12\mu_1-1},$$

where $B=a_{12\mu_1-\mu_2}+a_{12\mu_1-\mu_2+1}\delta+\ldots.+a_{12\mu_1-1}\delta^{\mu_1-1}$, and $\delta$ is a nilpotent such that $\delta^{\mu_1}=0$, $\lambda_{12i}\delta=\lambda_{12i+1}$. We need to notice then that we may also indicate in the same way a series of terms in $\lambda_{11i}$,

$$\lambda_{11}\delta G=a_{111}\lambda_{111}+a_{112}\lambda_{112}+\ldots.+a_{11\mu_1-1}\lambda_{11\mu_1-1},$$

where
$$G=a_{111}+a_{112}\delta+\ldots.+a_{11\mu_1-1}\delta^{\mu_1-2},$$

and
$$\lambda_{21}H=a_{210}\lambda_{210}+a_{211}\lambda_{211}+\ldots.+a_{21\mu_2-1}\lambda_{21\mu_2-1},$$

where
$$H=a_{210}+a_{211}\theta+\ldots.+a_{21\mu_2-1}\theta^{\mu_2-1}.$$

Hence in this notation, if we omit $\lambda$ and write only subscripts,

$$i=210+22\theta A+12\delta^{\mu_1-\mu_2}B, \quad j=111+22\theta C+12\delta^{\mu_1-\mu_2}E,$$

where

$$A(\theta)=a_0+a_1\theta+a_2\theta^2+\ldots.+a_{\mu_2-2}\theta^{\mu_2-2}, \quad B(\delta)=b_0+b_1\delta+b_2\delta^2+\ldots.+b_{\mu_2-1}\delta^{\mu_2-1},$$
$$C(\theta)=c_0+c_1\theta+c_2\theta^2+\ldots.+c_{\mu_2-2}\theta^{\mu_2-2}, \quad E(\delta)=e_0+e_1\delta+e_2\delta^2+\ldots.+e_{\mu_2-1}\delta^{\mu_2-1}.$$

If $\mu_1=\mu_2$, $b_0=b_1=0$ for 121 can not appear in $i$. We also use the same symbols for polynomials in $j$, the context always showing the use.

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* Shaw, *Trans. Am. Math. Soc.*, Vol. IV (1903), pp. 251–287.

The general form of the product $ji$ is necessarily

$$ji=ijC(j)+E(j) \text{ in which } e_0=e_1=0.$$

THEOREM I. *If $C\neq 1$ by a proper choice of $i$ the product $ji$ becomes $ji=ijC(j)$.*

PROOF: Let $X=x_1j+x_2j^2+\ldots.$, then

$$j(i-X)=ijC+E-jX=(i-X)jC+E+jX(C-1).$$

If we can determine $X$ so that $E+jX(C-1)=0$, we may set $i'=i-X$, whence $ji'=i'jC$, which is the form desired. We must have then

$$X=\frac{E}{j(1-C)}=\left(\frac{e_2j+e_3j^2+\ldots.+e_{\mu_1-1}j^{\mu_1-2}}{1-C}\right).$$

By Lemma I, $X$ is determinable provided the numerator does not start with a lower power of $j$ than the denominator. This is evidently the case whenever $1-C$ starts with a term free from $j$, but when $C=1+j^xC_1$, where $C_1$ starts with a term free from $j$, then the numerator must start with a higher term than $j^{x-1}$. To show that in this case the desired situation always exists we proceed as follows:

Let us abbreviate the writing by setting temporarily

$$i^2=ijA+B, \quad ji=ijC+j^wE_1,$$

where $E_1$ starts with a term free from $j$. Then we can easily see that if $P(j)$ is any polynomial in $j$,

$$P(j)i=iP(jC)+j^{w-1}E_1\frac{P(jC)-P(j)}{C-1} \text{ when } C\neq 1,$$

where the division is purely formal and always possible.

We have now $j\cdot i^2=jijA+jB=ij^2AC+$ terms in $j$. But we have by changing the association,

$$\begin{aligned} ji\cdot i&=ijCi+j^wE_1i=i\left[ijC\cdot C(jC)+j^{w-1}E_1\frac{jC\cdot C(jC)-jC}{C-1}\right]+ij^wC^wE_1(jC) \\ &=i^2jC\cdot C(jC)+ij^wCE_1\frac{C(jC)-1}{C-1}+ij^wC^wE_1(jC)+\ldots. \quad +\text{ terms in } j \\ &=ij^2AC\cdot C(jC)+ij^wC\left[E_1\frac{C(jC)-1}{C-1}+C^{w-1}E_1(jC)\right]+\ldots.. \end{aligned}$$

Since the terms in $ij^2$ must be equal, we have

$$ij^2\left[AC(-1+C(jC))+j^{w-2}C\left(E_1\frac{C(jC)-1}{C-1}+C^{w-1}E_1(jC)\right)\right]=0.$$

Now suppose that $C=1+j^xC_1$ where $C_1$ starts with a term free from $j$, then we have, since we may drop the factors that are invertible, by Lemma I.

$$ij^2\left[Aj^xC_1(jC)C^x+j^{w-2}\left(E_1\frac{C_1(jC)}{C_1}\cdot C^x+C^{w-1}E_1(jC)\right)\right]=0.$$

Now let $A=j^yA_1$ where $A_1$ starts with a term free from $j$, then the lowest terms from the two expressions that are multiplied by $i$ are

$$j^{x+y+2}a_{10}c_{10}+j^w2e_{10},$$

where $a_{10}$ is the first term of $A_1$, $c_{10}$ of $C_1$, and $e_{10}$ of $E_1$, none of which are zero. Either these two cancel each other, which can happen only if

$$x+y+2=w, \text{ and hence } x\leq w-2;$$

or else each is not lower than $\mu_2$. But we know that $x<\mu_2-1$, and hence if $\mu_2\leq w$ we have

$$x<\mu_2-1\leq w-1.$$

In any case then we must have $x\leq w-2$. Therefore whenever $C\neq 1$, $X(j)$ is determinable so that

$$ji=ijC(j).$$

Moreover, we have $X=j^{\mu_1-\mu_2}(X')$, and if $i'=i-X$, $i'j^{\mu_2}=ij^{\mu_2}-Xj^{\mu_2}=0$. Hence $i'$ is of the same character as $i$, and the transformation is permissible. In other words, whenever $C\neq 1$, we may take $E=0$ and write,

$$i=210+22\theta A+12\delta^{\mu_1-\mu_2}B, \quad j=111+22\theta C.$$

We proceed to investigate first the case $C=1$.

## IV. *Case I.* $C=1$.

If $C=1$ the generators have the form

$$i=210+22\theta A+12\delta^{\mu_1-\mu_2}B, \quad j=111+221+12\delta^{\mu_1-\mu_2}E.$$

Since, however, in this case

$$i^2=ijA+j^{\mu_1-\mu_2}B,$$

we may set $i'=i-\frac{1}{2}jA$, whence

$$ji'=ji+j^{\mu_1-\mu_2}E-\tfrac{1}{2}j^2A=i'j+j^{\mu_1-\mu_2}E,$$
$$i'^2=i^2-[ijA+\text{ terms in powers of } j]=\text{terms in powers of } j.$$

Consequently, we may treat the case as if $A=0$, until we derive the types. We then return to the equations in $i$, $j$. The generators now become

$$i=210+12\delta^{\mu_1-\mu_2}B, \quad j=111+221+12\delta^{\mu_1-\mu_2}E.$$

From these we form the following products:

$$ij=211+22\theta^{\mu_1-\mu_2}E+12\delta^{\mu_1-\mu_2+1}B, \quad ij^n=21n+22\theta^{\mu_1-\mu_2+n-1}nE+12\delta^{\mu_1-\mu_2+n}B,$$
$$ijG(j)=21\theta G(\theta)+22\theta^{\mu_1-\mu_2}E[g_0+2g_1\theta+3g_2\theta^2+\dots.]+12\delta^{\mu_1-\mu_2}BG(\delta),$$
$$j^2=112+222+12\delta^{\mu_1-\mu_2+1}2E, \quad j^n=11\delta^n+22\theta^n+12\delta^{\mu_1-\mu_2+n-1}nE,$$
$$ji=211+12\delta^{\mu_1-\mu_2+1}B+11\delta^{\mu_1-\mu_2}E,$$
$$j^{\mu_1-\mu_2}=11\delta^{\mu_1-\mu_2}+22\theta^{\mu_1-\mu_2}+12\delta^{2\mu_1-2\mu_2-1}(\mu_1-\mu_2)E,$$
$$j^{\mu_1-\mu_2}E(j)=11\delta^{\mu_1-\mu_2}E(\delta)+22\theta^{\mu_1-\mu_2}E(\delta)+12\delta^{2\mu_1-2\mu_2-1}E[(\mu_1-\mu_2)e_0+\dots.],$$
$$\therefore ji=ij+j^{\mu_1-\mu_2}E(j)-22\theta^{\mu_1-\mu_2}E$$
$$-12\delta^{2\mu_1-2\mu_2-1}E[(\mu_1-\mu_2)e_0+(\mu_1-\mu_2+1)e_1\delta+\dots.],$$

The last two terms must vanish, so we have

$$\theta^{\mu_1-\mu_2}E=0, \tag{1}$$
$$\delta^{2\mu_1-2\mu_2-1}E[(\mu_1-\mu_2)e_0+\dots.]=0, \tag{2}$$
$$i^2=22\theta^{\mu_1-\mu_2}B+11\delta^{\mu_1-\mu_2}B$$
$$=j^{\mu_1-\mu_2}B(j)+22\theta^{\mu_1-\mu_2}B-22\theta^{\mu_1-\mu_2}B-12\delta^{2\mu_1-2\mu_2-1}E[(\mu_1-\mu_2)b_0+\dots.].$$

This gives rise to the equation

$$\delta^{2\mu_1-2\mu_2-1}E[(\mu_1-\mu_2)b_0+(\mu_1-\mu_2+1)b_1\theta+\dots.]=0. \tag{3}$$

The solutions of these three equations are in three classes:

1. $\mu_1-\mu_2\geq\mu_2$, *i. e.*, $\mu_1\geq 2\mu_2$, $\mu_1-\mu_2=\mu_2+\alpha$, $E\neq 0$.
2. $\mu_1-\mu_2<\mu_2$, $E\neq 0$.
3. $E=0$.

For Class 1 equation (1) is satisfied. Equation (2) becomes

$$\delta^{2\mu_2+2\alpha-1}E[(\mu_2+\alpha)e_0+\dots.]=0, \text{ or } \delta^{\mu_1+\alpha-1}E[(\mu_2+\alpha)e_0+\dots.]=0.$$

This equation is satisfied unless $\alpha=0$ and then $E=\delta E_1$. Equation (3) becomes $\delta^{\mu_1+\alpha-1}E[(\mu_2+\alpha)b_0+\dots.]=0$, which is satisfied whenever (2) is.

For Class 2 let $E=\delta^v E_2$, where $e_{20}\neq 0$. Then (1) becomes $\theta^{\mu_1-\mu_2+v}E_2=0$.

$$\therefore \mu_1-\mu_2+v\geq\mu_2, \quad \mu_1+v\geq 2\mu_2.$$

Equation (2) becomes $\delta^{2\mu_1-2\mu_2+v-1}E_2[(\mu_1-\mu_2+v)e_{2v}.\dots.]=0$, which is satisfied unless $v=2\mu_2-\mu_1$, in which case $E_2=\delta E_3$ and $E=\delta^{v+1}E_3$, therefore $v>2\mu_2-\mu_1$. Equation (3) is satisfied with (2).

For Class 3 all three equations are satisfied. We have then for this case the following types:

11. $\mu_1>2\mu_2$, $\mu_1-\mu_2=\mu_2+\alpha$, $\alpha>0$, $E\neq 0$.
12. $\mu_1=2\mu_2$, $E=\delta E_1$, $E_1\neq 0$.
2. $\mu_1-\mu_2<\mu_2$, $E=\delta^v E_3$, $E_3\neq 0$, $v>2\mu_2-\mu$.
3. $E=0$.

These types are now written out as follows:

11. $\mu_1 > 2\mu_2$, $\mu_1 - \mu_2 = \mu_2 + \alpha$, $\alpha > 0$, $E \neq 0$. The generators for this type are

$$i = 210 + 12\delta^{\mu_2+\alpha}B, \quad j = 111 + 221 + 12\delta^{\mu_2+\alpha}E.$$

From these we get the following products:

$$ij = 211 + 12\delta^{\mu_2+\alpha+1}B, \quad j^2 = 112 + 222 + 12\delta^{\mu_2+\alpha+1}2E,$$
$$j^n = 11n + 22n + 12\delta^{\mu_2+\alpha+n-1}nE,$$
$$ji = 211 + 12\delta^{\mu_2+\alpha+1}B + 11\delta^{\mu_2+\alpha}E = ij + j^{\mu_2+\alpha}E(j), \quad i^2 = 11\delta^{\mu_2+\alpha}B = j^{\mu_2+\alpha}B(j).$$

12. $\mu_1 = 2\mu_2$, $E = \delta E_1$, $E_1 \neq 0$:

$$i = 210 + 12\delta^{\mu_1}B, \quad j = 111 + 221 + 12\delta^{\mu_2+1}E_1, \quad ij = 211 + 12\delta^{\mu_2+1}B,$$
$$j^n = 11n + 22n + 12\delta^{\mu_2+n}nE_1,$$
$$ji = 211 + 12\delta^{\mu_2+1}B + 11\delta^{\mu_2+1}E_1 = ij + j^{\mu_2+1}E_1(j), \quad i^2 = 11\delta^{\mu_2}B = j^{\mu_2}B(j).$$

2. $\mu_1 - \mu_2 < \mu_2$, $E = \delta^v E_3$, $E_3 \neq 0$, $v > 2\mu_2 - \mu_1$:

$$i = 210 + 12\delta^{\mu_1-\mu_2}B, \quad j = 111 + 221 + 12\delta^{\mu_1-\mu_2+v}E_3,$$
$$ij = 211 + 12\delta^{\mu_1-\mu_2+1}B, \quad j^n = 11n + 22n + 12\delta^{\mu_1-\mu_2+v+n-1}nE_3,$$
$$ji = 211 + 12\delta^{\mu_1-\mu_2+1}B + 11\delta^{\mu_1-\mu_2+v}E_3 = ij + j^{\mu_1-\mu_2+v}E_3(j),$$
$$i^2 = 22\theta^{\mu_1-\mu_2}B + 11\delta^{\mu_1-\mu_2}B = j^{\mu_1-\mu_2}B(j).$$

3. $E = 0$:

$$i = 210 + 12\delta^{\mu_1-\mu_2}B, \quad j = 111 + 221, \quad ij = 211 + 12\delta^{\mu_1-\mu_2+1}B,$$
$$ji = 211 + 12\delta^{\mu_1-\mu_2+1}B = ij, \quad j^n = 11n + 22n,$$
$$i^2 = 11\delta^{\mu_1-\mu_2}B + 22\theta^{\mu_1-\mu_2}B = j^{\mu_1-\mu_2}B(j).$$

| Type | $ji' =$ | $(i')^2 =$ |
|---|---|---|
| 1 | $i'j$, | $j^{\mu_1-\mu_2}B(j)$, |
| 2 | $i'j + j^{\mu_2+1}E_1(j)$, | $j^{\mu_2}B(j)$, |
| 3 | $i'j + j^{\mu_2+\alpha}E(j)$, | $j^{\mu_2+\alpha}B(j)$, |
| 4 | $i'j + j^{\mu_1-\mu_2+v}E_3(j)$, | $j^{\mu_1-\mu_2}B(j)$, |

or in terms of $i$ and $j$, instead of $i'$ and $j$.

| Type | $ji =$ | $i^2 =$ |
|---|---|---|
| 1 | $ij$, | $ijA + \frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)' - \frac{1}{4}j^2A^2 + j^{\mu_1-\mu_2}B(j)$, |
| 2 | $ij + j^{\mu_2+1}E_1(j)$, | $ijA + \frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)' - \frac{1}{4}j^2A^2 + j^{\mu_2}B(j)$, |
| 3 | $ij + j^{\mu_2+\alpha}E(j)$, | $ijA + \frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)' - \frac{1}{4}j^2A^2 + j^{\mu_1-\mu_2}B(j)$, |
| 4 | $ij + j^{\mu_1-\mu_2+v}E_3(j)$, | $ijA + \frac{1}{2}j^{\mu_1-\mu_2}E(j)(jA)' - \frac{1}{4}j^2A^2 + j^{\mu_1-\mu_2}B(j)$, |

when $(jA)'$ is the formal derivative as to $j$ of the polynomial $jA$, and $A$ is arbitrary. In case $A$ starts with a term of order $\mu_1 - \mu_2 - 1$ or higher, $i'$ may be used for $i$.

V. *Case II.* $A=0,\ C\neq 1,\ E=0.$

If $A$ vanishes we have the defining equations:

$$i = 210+12\delta^{\mu_1-\mu_2}B,\quad j=111+22\theta C.$$

Then $ij^s=21s+12\delta^{\mu_1-\mu_2+s}BC^s$, and $iP(j)=21P(\theta)+12\delta^{\mu_1-\mu_2}BP(\delta C)$.

Again $ji = 21\theta C+12\delta^{\mu_1-\mu_2+1}B=ijC(j)-12\delta^{\mu_1-\mu_2+1}BC\cdot C(\delta C)+12\delta^{\mu_1-\mu_2+1}B$

$$\therefore\ \delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0. \tag{1}$$

Also $j^2 = 112+22\theta^2C^2$, ....., whence $P(j)=11P+22P(\theta C)$,

$i^2 = 11\delta^{\mu_1-\mu_2}B+22\theta^{\mu_1-\mu_2}B,$

but $j^{\mu_1-\mu_2}=11\delta^{\mu_1-\mu_2}+22\theta^{\mu_1-\mu_2}C^{\mu_1-\mu_2}$, so $i^2=j^{\mu_1-\mu_2}B(j)$,

$$\therefore\ \theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0. \tag{2}$$

Let $C$ be written $C=c_0+\theta^tC_1$ where $c_0$ may be 0, but $c_{10}\neq 0$ and $0<t<\mu_2-1$. Substituting in equation (1)

$$\delta^{\mu_1-\mu_2+1}B[1-(c_0+\theta^tC_1)(c_0+\theta^tC^tC_1(\theta C))]=0,$$

or $$\delta^{\mu_1-\mu_2+1}B[1-c_0^2-\theta^tc_0C_1-c_0\theta^tC^tC_1(\theta C)]=0.$$

By applying Lemma II we have either

11. $c_0^2\neq 1$ and $B=\delta^{\mu_2-1}b_{\mu_2-1}$,

or 12. $c_0^2=1$ and $B=\delta^{\mu_2-t-1}B_1$.

Now substituting in equation (2) the results just found, we have for 11:

$$\theta^{\mu_1-1}b_{\mu_2-1}[1-(c_0+\theta^tC_1)^{\mu_1-1}]=0.$$

Therefore by Lemma II,

111, $b_{\mu_2-1}=0$, and $B=0$; or 112, $c_0^{\mu_1-1}=1$, $b_{\mu_2-1}\neq 0$.

For the case 12, $$\theta^{\mu_1-t-1}[B_1-C^{\mu_1-t-1}B_1(\theta C)]=0.$$

Hence either 121, for $c_0=1$, $\theta^{\mu_1-1}b_{10}(\mu_1-t-1)C_1=0$, for which we have either

1211, $\mu_1>\mu_2$, $B=\theta^{\mu_2-t-1}B_1$,

or 1212, $\mu_1=\mu_2$, $B=\theta^{\mu_2-t}B_2$,

or 122, for $c_0=-1$, $\theta^{\mu_1-t-1}[B_1-(-1+\theta^tC_1)^{\mu_1-t-1}B_1(\theta C)]=0$,

for which we have

1221, $\mu_1-1$ is even, $B_1=\theta^tB_3$, *i. e.*, $B=\theta^{\mu_2-1}b_{\mu_2-1}$, $b_{\mu_2-1}\neq 0$,

or 1222, $B_1=\theta^{t+1}B_4$, *i. e.*, $B=0$,

or 1223, $\mu_1-t-1\geq\mu_2$, $B_1\neq 0$.

Consequently when $A=0$ the following seven types arise:

$$A=0, \quad C=c_0+\theta^t C_1,$$

111. $c_0^2 \neq 1$, $B=0$.
112. $c_0^{\mu_1-1}=1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$.
1211. $c_0=1$, $B=\delta^{\mu_2-t-1}B_1$, $\mu_1>\mu_2$.
1212. $c_0=1$, $B=\delta^{\mu_2-t}B_2$, $\mu_1=\mu_2$.
1221. $c_0=-1$, $\mu_1-1$ is even, $B=\delta^{\mu_2-1}b_{\mu_2-1}$.
1222. $c_0=-1$, $B=0$.
1223. $c_0=-1$, $\mu_1-t-1\geq\mu_2$, $B=\delta^{\mu_2-t-1}B_1$, $B_1\neq 0$.

We now work out these seven types obtaining the expressions for the products $ji$ and $i^2$.

111. $C=c_0+\theta^t C_1$, $c_0^2\neq 1$, $B=0$. For this the generators have the form
$i=210$, $j=111+22\theta C$.

Then $ij=211$, $ji=21\theta C=ijC(j)$, and $i^2=0$.

112. $c_0^{\mu_1-1}=1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$.

Now $i=210+12\delta^{\mu_1-1}b_{\mu_2-1}$, $j=111+22\theta C$, $ij=211$, $ji=21\theta C=ijC(j)$,
$i^2=11\delta^{\mu_1-1}b_{\mu_1-1}+22\theta^{\mu_1-1}b_{\mu_2-1}$, $j^n=11\delta^n+22\theta^n C^n$, $\therefore\ i^2=j^{\mu_1-1}b_{\mu_2-1}$.

1211. $c_0=1$, $B=\delta^{\mu_2-t-1}B_1$, $\mu_1>\mu_2$.
$i=210+12\delta^{\mu_1-t-1}B_1$, $j=111+221+22\theta^{t+1}C_1$, $ij^n=21n+12\delta^{\mu_1-t+n-1}B_1$,
$ji=211+12\delta^{\mu_1-t}B_1+21\theta^{t+1}C_1=ij+ij^{t+1}C_1(j)$,
$j^n=11n+22\theta^n C^n=11n+22\theta^n(1+\theta^t C_1)^n$,
$i^2=11\delta^{\mu_1-t-1}B_1+22\theta^{\mu_1-t-1}B_1=j^{\mu_1-t-1}B_1(j)$.

1212. $c_0=1$, $B=\delta^{\mu_2-2}B_2$, $\mu_1=\mu_2$.
$i=210+12\delta^{\mu_1-t}B_2$, $j=111+221+22\theta^{t+1}C_1=111+22\theta C$,
$ij^n=21n+12\delta^{\mu_1-t+n}B_2$, $ji=211+21\theta^{t+1}C_1+12\delta^{\mu_1-t+1}B_2=ij+ij^{t+1}C_1(j)$,
$i^2=11\delta^{\mu_1-t}B_2+22\theta^{\mu_1-t}B_2=j^{\mu_1-t}B_2(j)$.

1221. $\mu_1-1$ is even, $B=\delta^{\mu_2-1}b$, $c_0=-1$.
$i=210+12\delta^{\mu_1-1}b$, $j=111+221(-1)+22\theta^{t+1}C_1$, $ij=211$,
$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j)$,
$j^n=11n+22\theta^n(-1+\theta^t C_1)^n$, $i^2=11\delta^{\mu_1-1}b+22\theta^{\mu_1-1}b=j^{\mu_1-1}b$.

1222. $c_0=-1$, $B=0$.
$i=210$, $j=111+221(-1)+22\theta^{t+1}C_1$, $ij=211$,
$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j)$, $i^2=0$.

1223. $c_0=-1$, $\mu_1-t-1\geq\mu_2$, $B=\delta^{\mu_1-t-1}B_1$, $B_1\neq 0$.

$i=210+12\delta^{\mu_1-t-1}B_1$, $j=111+221(-1)+22\theta^{t+1}C_1$,

$ij^n=21n+12\delta^{\mu_1-t+n-1}(-1)^nB_1$,

$ji=211(-1)+21\theta^{t+1}C_1+12\delta^{\mu_1-t}B_1=-ij+ij^{t+1}C_1(j)$,

$i^2=11\delta^{\mu_1-t-1}B_1=j^{\mu_1-t-1}B_1(j)$.

| Type | $ji=$ | $i^2=$ | |
|---|---|---|---|
| 1 | $ij[c_0+j^tC_1(j)]$, | 0, | $c_0^2\neq 1$, |
| 2 | $ij[c_0+j^tC_1(j)]$, | $j^{\mu_1-1}b_{\mu_2-1}$, | $c_0^2\neq 1$, $b_{\mu_2-1}\neq 0$, |
| 3 | $ij+ij^{t+1}C_1(j)$, | $j^{\mu_1-t-1}B_1(j)$, | $\mu_1>\mu_2$, |
| 4 | $ij+ij^{t+1}C_1(j)$, | $j^{\mu_1-t}B_2(j)$, | $\mu_1=\mu_2$, |
| 5 | $-ij+ij^{t+1}C_1(j)$, | $j^{\mu_1-1}b$, | $\mu_1$ is odd, $b\neq 0$, |
| 6 | $-ij+ij^{t+1}C_1(j)$, | 0, | |
| 7 | $-ij+ij^{t+1}C_1(j)$, | $j^{\mu_1-t-1}B_1(j)$, | $\mu_1-t-1\geq\mu_2$, $B_1\neq 0$. |

## VI. *Case III.* $A\neq 0$, $C\neq 1$, $E=0$.

In this case, which is the most general, we first form the following products from the generators:

$i=210+22\theta A+12\delta^{\mu_1-\mu_2}B$, $j=111+22\theta C$,

$ij^r=21r+22\theta^{r+1}AC^r+12\delta^{\mu_1-\mu_2+r}BC^r$, $r<\mu_2-1$,

$ij^{\mu_2-1}=21\theta^{\mu_2-1}+12\delta^{\mu_1-1}BC^{\mu_2-1}$,

$iP(j)=210P(\theta)+22\theta AP(\theta C)+12\delta^{\mu_1-\mu_2}BP(\delta C)$, where $P(j)$ is a polynomial in $j$.

$j^s=11s+22\theta^sC^s$, $s<\mu_2$, $j^s=11\theta^s$, $\mu_1>s\geq\mu_2$,

$P(j)=11P(\delta)+22P(\theta C)$,

$ji=21\theta C+22\theta^2AC+12\delta^{\mu_1-\mu_2+1}B$

$=ijC(j)+22\theta^2AC+12\delta^{\mu_1-\mu_2+1}B-22\theta^2AC\cdot C(\theta C)-12\delta^{\mu_1-\mu_2+1}BCC(\theta C)$,

$P(j)\cdot i\theta=210P(\theta C)+22\theta AP(\theta C)+12\delta^{\mu_1-\mu_2}BP(\delta)=i\cdot P(jC)$,

$i^2=21\theta A+22\theta^2A^2+22\theta^{\mu_1-\mu_2}B+11\delta^{\mu_1-\mu_2}B+12\delta^{\mu_1-\mu_1+1}AB$

$=ijA(j)+j^{\mu_1-\mu_2}B(j)+22\theta^2A^2+22\theta^{\mu_1-\mu_2}B+12\delta^{\mu_1-\mu_2+1}AB$

$-22\theta^2ACA(\theta C)-22\theta^{\mu_1-\mu_2}C^{\mu_1-\mu_2}B(\theta C)-12\delta^{\mu_1-\mu_2}BCA(\theta C)$.

Since the terms in 22() and 12() must vanish in both the products $ji$ and $i^2$, we have four equations:

$$\left.\begin{array}{ll}(1) & \theta^2AC[1-C(\theta C)]=0,\\ (2) & \delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0,\\ (3) & \delta^{\mu_1-\mu_2+1}B[A-C\cdot A(\delta C)]=0,\\ (4) & \theta^2A[A-C\cdot A(\theta C)]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0.\end{array}\right\}\quad (A)$$

Before reducing these we need a second theorem which we proceed to deduce.

If $A(j)=j^sA_1(j)$, $0\leq s<\mu_1-1$, then $i^2=ij^{s+1}A_1(j)+j^{\mu_1-\mu_2}B(j)$, where $a_{10}\neq 0$. By Lemma III we may now express a polynomial in $j$ in terms of one in $jC$ if $C=c_0+jC_1$. Let $A_1(j)=A_1'(jC)$ and determine $i'$ from $i=i'A_1'(j)$ where $a_{10}\neq 0$, then

$$i^2=i'A_1'i'A_1'=i'j^{s+1}A_1(j)\cdot A_1'(j)+j^{\mu_1-\mu_2}B(j).$$

But using the formula $P(j)i=iP(jC)$,

$$A_1'(j)\cdot i=iA_1'(jC)=iA_1(j),$$
$$A_1'(j)\cdot i'=iA_1(j)A_1'^{-1}(j)=i'A_1(j).$$

Thence $\quad i'A_1'i'A_1'=i'^2A_1A_1'=i'j^{s+1}A_1(j)A_1'(j)+j^{\mu_1-\mu_2}B(j).$

That is $\quad i'^2=i'j^{s+1}+j^{\mu_1-\mu_2}B(j)A_1^{-1}(j)A_1'^{-1}(j)=i'j^{s+1}+j^{\mu_1-\mu_2}B'.$

THEOREM II. *If $C(j)=c_0+jC_1(j)$, where $c_0\neq 0$ then $j=P(jC)$ and $A_1'(j)$ is known, and invertible. Hence we may choose $i$ so that the term $ij^{s+1}A_1(j)$ becomes simply $ij^{s+1}$, that is, we may take $A_1=1$.*

As a consequence of Theorem II we have two sub-cases,

$\text{III}_1$. $A(j)=j^s$, $C(j)=c_0+jC_1(j)$, $c_0\neq 0$.
$\text{III}_2$. $C(j)=j^tC_1(j)$, $t>0$, $c_{10}\neq 0$.

Case $\text{III}_1$. The generators may now be written

$$i=210+22\theta^{s+1}+12\delta^{\mu_1-\mu_2}B,\quad j=111+22\theta C,$$

and we shall have

$$ji=ijC(j),\quad i^2=ij^{s+1}+j^{\mu_1-\mu_2}B(j),$$

where the following four equations, which are the reduced forms of equations (A), must be simultaneously satisfied:

$$\left.\begin{array}{ll}(1)' & \theta^{s+2}C[1-C(\theta C)]=0,\\ (2)' & \delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0,\\ (3)' & \delta^{\mu_1-\mu_2+s+1}[1-C^{s+1}]=0,\\ (4)' & \theta^{2s+2}[1-C^{s+1}]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0.\end{array}\right\}\quad (A)'$$

Consider equation (1)',

$$\theta^{s+2}C[1-C(\theta C)]=0.$$

This leads to two classes 3 and 4.

3 $s+2=\mu_2$, $C=c_0+\theta^tC_1$, $c_0\neq 0$, $c_{10}\neq 0$.

4 $s+2<\mu_2$, then since $c_0\neq 0$, $1-C(\theta C)=-\theta^{\mu_2-s-2+r}C_2$, or $C(\theta C)=1+\theta^{\mu_2-s-2+r}C_2$, $c_{20}\neq 0$. $\therefore$ $c_0+\theta^tC_1(\theta C)C^t=1+\theta^{\mu_2-s-2+r}C_2$,

or $\quad c_0+\theta^t[c_{10}+\theta C\cdot C_3(\theta C)][c_0^t+\theta^tC_4]=1+\theta^{\mu_2-s-2+r}C_2,$

or $\quad c_0+\theta^tc_{10}c_0^t+\theta^{t+1}C_5=1+\theta^{\mu_2-s-2+r}C_2.$

Therefore $c_0=1$ and $t=\mu_2-s-2+r$ since $c_0^t\neq 0\neq c_{10}$ and $C(\theta C)=1+\theta^t C_2$, $t<\mu_2$. Consider next equation (3)′,

$$\delta^{\mu_1-\mu_2+s+1}B[1-C^{s+1}]=0.$$

For class 3, $s+2=\mu_2$, $C=c_0+\theta^t C_1$, $c_0\neq 0$, $c_{10}\neq 0$ this becomes

$$\delta^{\mu_1-1}b_0[1-C^{\mu_2-1}]=0,$$

from which we have either

31 $$b_0=0,\quad B=\delta B_1,\quad c_0^{\mu_2-1}\neq 1,$$

or 32 $$c_0^{\mu_2-1}=1.$$

For class 4, $s+2<\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$ equation (3)′ becomes

$$\delta^{\mu_1-\mu_2+s+1}B[1-(1+\delta^{\mu_2-s-2+r}C_1)^{s+1}]=0,$$
$$\delta^{\mu_1-1+r}B[-C_1(s+1)+\ldots.]=0,\quad \delta^{\mu_1-1+r}B=0.$$

This is satisfied unless $r=0$, then $B=\delta B_1$. Consider now equation (2)′,

$$\delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0.$$

For 31 $s+2=\mu_2$, $c_0^{\mu_2-1}\neq 1$, $C=c_0+\delta^t C_1$, $c_0\neq 0\neq c_{10}$. Put $B=\delta^v B_2$, $b_{20}\neq 0$. Then $\delta^{\mu_1-\mu_2+v+1}B_2[1-C\cdot C(\delta C)]=0=\delta^{\mu_1-\mu_2+v+1}[1-C\cdot C(\delta C)]$.

311 If $c_0^2\neq 1$ then $v\geq\mu_2-1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$, or $B=0$ if $b_{\mu_2-1}=0$.

312 If $c_0^2=1$, $c_0^{\mu_2-1}\neq 1$, that is $c_0=-1$ and $\mu_2-1$ is odd, then

$$\mu_1-\mu_2+v+t+1\geq\mu_1.\quad \therefore\ v\geq\mu_2-t-1.$$

For 32 $c_0^{\mu_2-1}=1$, then $\delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0$,

$$\delta^{\mu_1-\mu_2+1}B[1-(c_0+\delta^t C_1)(c_0+\delta^t C^t C_1(\delta C)]=0.$$

321 If $c_0^2\neq 1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$.

322 If $c_0^2=1$, $B=\delta^{\mu_2-t-1}B_3$, and if $c_0=-1$, $\mu_2-1$ is even, that is $\mu_2$ is odd.

For Class 4, $s+2<\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$, equation (2)′ becomes

$$\delta^{\mu_1-\mu_2+1}B[1-C\cdot C(\delta C)]=0,\quad \delta^{\mu_1-\mu_2+1}B[1-(1+\delta^t C_1)(1+\delta^t C^t C_1(\delta C)]=0,$$
$$\therefore\ \delta^{\mu_1-\mu_2+1+t}B=0,\text{ which is }\delta^{\mu_1-s-1+r}B=0.\quad \therefore\ B=\delta^{s+1-r}B_4.$$

We consider finally equation (4)′,

$$\theta^{2s+2}[1-C^{s+1}]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0.$$

For 311 $c_0^2\neq 1$, $B=\delta^{\mu_2-1}b$ where $b$ may be 0, $s+2=\mu_2$, $c_0^{\mu_2-1}\neq 1$, $c_0\neq 0\neq c_{10}$, then $\theta^{\mu_1-1}b[1-(c_0+\theta^t C_1)^{\mu_1-1}]=0$, which is satisfied unless $\mu_1=\mu_2$ since $\theta^{\mu_2}=0$. Therefore we have also

3111 $\mu_1=\mu_2$, $b=b^*_{\mu_2-1}=0$, *i. e.*, $v>\mu_2-1$, and

3112 $\mu_1=\mu_2$, $c_0^{\mu_1-1}=1$, but this is impossible since it is a contradiction to 311.

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*We write $b$ in place of $b_{\mu_2-1}$ in what follows.

For 312 $c_0=-1$, $\mu_2-1$ is odd, $B=\theta^v B_2$, $b_{20}\neq 0$, $v>\mu_2-t-1$, equation (4)′ becomes

$$\theta^{\mu_1-\mu_2+v}[B_2-(-1+\theta C_1)^{\mu_1-\mu_2+v}B_2(\theta C)]=0.$$

We have then either

3121, $\mu_1-\mu_2+v$ is odd, and $\mu_1-\mu_2+v\geq\mu_2$, *i. e.*, $v\geq 2\mu_2-\mu_1$, or
3122, $\mu_1-\mu_2+v$ is even.

Now $\theta^{\mu_1-\mu_2+v}[B_2-B_2(\theta C)-\ldots.]=0$

becomes $\theta^{\mu_1-\mu_2+v+1}[b_{21}(1-C)+\ldots.]=0,$

and $\mu_1-\mu_2+v+1\geq\mu_2$, *i. e.*, $v\geq 2\mu_2-\mu_1-1$.

For 321 $c_0^2\neq 1$, $B=\delta^{\mu_2-1}b_{\mu_2-1}$, $c_0^{\mu_2-1}=1$, equation (4)′ becomes

$$\theta^{\mu_1-1}b_{\mu_2-1}[1-(c_0+\theta^t C_1)^{\mu_1-1}]=0$$

which is satisfied since $\theta^{\mu_2}=0$. If $\mu_1>\mu_2$, $\theta^{\mu_1-1}=0$. If $\mu_1=\mu_2$, the [] causes the expression to vanish.

For 322 $c_0^2=1$, $B=\delta^{\mu_2-t-1}B_3$, $\mu_2-1$ is even if $c_0=-1$ and (4)′ becomes $\theta^{\mu_1-t-1}[B_3-(c_0+\theta^t C_1)^{\mu_1-t-1}B_3(\theta C)]=0$, for which we have

3221, $c_0=1$, $\theta^{\mu_1-t-1}[b_{31}\theta(1-C)+\ldots.-\theta^t C_1 B_3(\theta C)-\ldots.]=0$,
$\therefore$ $\theta^{\mu_1-1}B_3(\theta C)=0$, which is satisfied unless $\mu_1=\mu_2$. Then
32211, $\mu_1>\mu_2$.
32212, $\mu_1=\mu_2$, $B_3(\theta C)=\theta B_5$, *i. e.*, $B=\theta^{\mu_2-t}B$.
3222, $c_0=-1$, $\mu_2-1$ is even, then

$$\theta^{\mu_1-t-1}[B_3-(-1)^{\mu_1-t-1}B_3(\theta C)-(-1)^{\mu_1-t}\theta^t C_1 B_3(\theta C)+\ldots.]=0.$$

32221. If $\mu_1-t-1$ is even, $\theta^{\mu_1-t-1}[b_{31}\theta(1-C)+\ldots.]=0$.
$\therefore$ $B_3=\theta^t B_6$, which makes $B=\theta^{\mu_2-1}b_{\mu_2-1}$.
32222. If $\mu_1-t-1$ is odd, $B_3=\theta^{t+1}B_7$, which makes $B=0$.

For Class 4 $s+2<\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$, $B=\theta^{s+1-r}B_4$.
Equation (4)′ becomes

$$\theta^{2s+2}[1-C^{s+1}]+\theta^{\mu_1-\mu_2}[B-C^{\mu_1-\mu_2}B(\theta C)]=0,$$
$$\theta^{\mu_1-\mu_2+s+1-r}[B_4-(1+\theta^t C_1)^{\mu_1-\mu_2+s+1-r}B_4(\theta C)]=0,$$
$$\theta^{\mu_1-\mu_2+s+1-r}[b_{41}\theta(1-C)+\ldots.+\theta^t C_1 B_4(\theta C)+\ldots.]=0,$$
$$\theta^{\mu_1-1}[b_{41}\theta+b_{42}\theta^2(1+C)+\ldots.+C_1 B_4(\theta C)+\ldots.]=0,$$
$\therefore$ $\theta^{\mu_1-1}B_4(\theta C)=0$, for which we have either

41, $\mu_1>\mu_2$, or
42, $\mu_1=\mu_2$, $B_4=\theta B_6$, *i. e.*, $B=\theta^{s+2-r}B_6$.

We have therefore in Case $\mathrm{III}_1$ the following eleven types:

$s+2=\mu_2$ {
311. $\mu_1>\mu_2$, $B=\theta^{\mu_2-1}b_{\mu_2-1}$, $c_0^{\mu_2-1}\neq 1$, $c_0^2\neq 1$.
3111. $\mu_1=\mu_2$, $B=0$, $c_0^2\neq 1$, $c_0^{\mu_2-1}\neq 1$.
3121. $c_0=-1$, $c_0^{\mu_2-1}\neq 1$, $\mu_2-1$ is odd, $B=\theta^v B_2$, $\mu_1-\mu_2+v$ is odd, $v\geq 2\mu_2-\mu_1$, $v>\mu_2-t-1$, $\mu_1-\mu_2\geq t+1$.
3122. $c_0=-1$, $c_0^{\mu_2-1}\neq 1$, $\mu_2-1$ is odd, $B=\theta^v B_2$, $\mu_1-\mu_2+v$ is even, $v\geq 2\mu_2-\mu_1-1$.
321. $c_0^2\neq 1$, $c_0^{\mu_2-1}=1$, $B=\theta^{\mu_2-1}b_{\mu_2-1}$.
32211. $\mu_1>\mu_2$, $c_0=1$, $B=\theta^{\mu_2-t-1}B_3$.
32212. $\mu_1=\mu_2$, $c_0=1$, $B=\theta^{\mu_2-t}B_5$.
32221. $c_0=-1$, $\mu_2-1$ is even, $\mu_1-t-1$ is even, $B=\theta^{\mu_2-1}b_{\mu_2-1}$.
32222. $c_0=-1$, $\mu_2-1$ is even, $\mu_1-t-1$ is odd, $B=0$.

$s+2>\mu_2$ {
41. $\mu_1>\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$, $B=\theta^{s+1-r}B_4$.
42. $\mu_1=\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$, $B=\theta^{s+2-r}B_6$.

These particular types are written out as follows:

311. $s+2=\mu_2$, $\mu_1>\mu_2$, $c_0^2\neq 1$, $B=\theta^{\mu_2-1}b_{\mu_2-1}$, $C=c_0+\theta^t C_1$.

$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-1}b$, $j=111+22\theta C$.

Then $ij=211$, $ji=21\theta C=ijC(j)=c_0 ij+ij^{t+1}C_1(j)$,
$i^2=21\theta^{\mu_2-1}+11\delta^{\mu_1-1}b=ij^{\mu_2-1}+j^{\mu_1-1}b$.

3111. $s+2=\mu_2$, $\mu_1=\mu_2$, $B=0$, $c_0^2\neq 1$, $c^{\mu_2-1}\neq 1$.

$i=210+22\theta^{\mu_2-1}$, $j=111+22\theta C$.

Then $ij=211$, $ji=21\theta C=ijC(j)=c_0 ij+ij^{t+1}C_1(j)$, $i^2=21\theta^{\mu_2-1}=ij^{\mu_2-1}$.

3121. $c_0=-1$, $c_0^{\mu_2-1}\neq 1$, $\mu_2-1$ is odd, $B=\theta^v B_2$, $\mu_1-\mu_2+v$ is odd, $v\geq 2\mu_2-\mu_1$, $v>\mu_2-t-1$, $\mu_1-\mu_2\geq t+1$.

$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-\mu_2+v}B_2$,
$j=111+22(-1)+22\theta^{t+1}C_1=111+22\theta C$, $ij^n=21n+12\delta^{\mu_1-\mu_2+v+n}(-1)^n B_2$,
$ji=211(-1)+12\delta^{\mu_1-\mu_2+v+1}B_2+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j)$,
$j^n=11n+22\theta^n C^n=11n+22\theta^n(-1+\theta^t C_1)^n$, $j^{\mu_1-\mu_2+v}=11\delta^{\mu_1-\mu_2+v}$,
$i^2=21\theta^{\mu_2-1}+22\theta^{\mu_1-\mu_2+v}B_2+12\delta^{\mu_1+v-1}B_2+11\delta^{\mu_1-\mu_2+v}B_2$
$=ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j)$.

3122. $c_0=-1$, $c_0^{\mu_2-1}\neq 1$, $\mu_2-1$ is odd, $B=\theta^v B_2$, $\mu_1-\mu_2+v$ is even, $v\geq 2\mu_2-\mu_1-1$, $v>\mu_2-t-1$.

$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-\mu_2+v}B_2$,
$j=111+221(-1)+22\theta^{t+1}C_1=111+22\theta C$, $ij^n=21n+12\delta^{\mu_1-\mu_2+v+n}(-1)^n B_2$,
$ji=211(-1)+21\theta^{t+1}C_1+12\delta^{\mu_1-\mu_2+v+1}B_2=-ij+ij^{t+1}C_1(j)$,
$j^n=11n+22\theta^n(-1+\theta^t C_1)^n$, $j^{\mu_1-\mu_2+v}=11\delta^{\mu_1-\mu_2+v}$,
$i^2=21\theta^{\mu_2-1}+11\delta^{\mu_1-\mu_2+v}B_2+12\delta^{\mu_1+v-1}B_2=ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j)$.

321. $c_0^2 \neq 1$, $c_0^{\mu_2-1}=1$, $B=\theta^{\mu_2-1}b$, $s+2=\mu_2$.

$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-1}b$, $j=111+22\theta C$, $ij=211$,

$ji=21\theta C=ijC(j)$, $i^2=21\theta^{\mu_2-1}+22\theta^{\mu_1-1}b+11\delta^{\mu_1-1}b=ij^{\mu_2-1}+j^{\mu_1-1}b$.

32211. $c_0=1$, $\mu_1>\mu_2$, $B=\theta^{\mu_2-t-1}B_3$, $s+2=\mu_2$.

$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-t-1}B_3$,

$j=111+221+22\theta^{t+1}C_1$, $ij^n=21n+12\delta^{\mu_1-t+n-1}B_3$,

$ji=211+12\delta^{\mu_1-t}B_3+21\theta^{t+1}C_1=ij+ij^{t+1}C_1(j)$, $j^n=11n+22\theta^n(1+\theta^tC_1)^n$,

$i^2=21\theta^{\mu_2-1}+22\theta^{\mu_1-t-1}B_3+11\delta^{\mu_1-t-1}B_3+12\delta^{\mu_1-\mu_2-t-2}B_3$

$=ij^{\mu_2-1}-12\delta^{\mu_1-\mu_2-t-2}B_3+j^{\mu_1-t-1}B_3-22\theta^{\mu_1-t-1}B_3$

$+12\delta^{\mu_1-\mu_2-t-2}B_3+22\theta^{\mu_1-t-1}B_3=ij^{\mu_2-1}+j^{\mu_1-t-1}B_3(j)$.

32212. $c_0=1$, $\mu_1=\mu_2$, $B=\theta^{\mu_2-t}B_5$, $s+2=\mu_2$.

$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-t}B_5$, $j=111+221+22\theta^{t+1}C_1$,

$ij^n=21n+12\delta^{\mu_1-t+n}B_5$, $ji=211+21\theta^{t+1}C_1+12\delta^{\mu_1-t+1}B_5=ij+ij^{t+1}C_1(j)$,

$j^n=11n+22\theta^n(1+\theta^tC_1)^n$, $i^2=22\theta^{\mu_1-t}B_5+21\theta^{\mu_2-1}+11\delta^{\mu_1-t}B_5$,

$i^2=ij^{\mu_2-1}+j^{\mu_1-t}B_5(j)$.

32221. $c_0=-1$, $\mu_2-1$ is even, $\mu_1-t-1$ is even, $B=\theta^{\mu_2-1}b$, $s+2=\mu_2$.

$i=210+22\theta^{\mu_2-1}+12\delta^{\mu_1-1}b$, $j=111+221(-1)+22\theta^{t+1}C_1$, $ij=211$,

$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j)$, $j^n=11n+22\theta^n(-1+\theta^tC_1)^n$,

$i^2=21\theta^{\mu_2-1}+22\theta^{\mu_1-1}b+11\delta^{\mu_1-1}b$

$=ij^{\mu_2-1}+j^{\mu_1-1}b-22\theta^{\mu_1-1}b(-1)^{\mu_1-1}+22\theta^{\mu_1-1}b$.

Since $\theta^{\mu_1-1}=0$, unless $\mu_1=\mu_2$, these last two terms vanish separately. If $\mu_1=\mu_2$ they destroy each other since $(-1)^{\mu_2-1}=1$.

32222. $c_0=-1$, $\mu_2-1$ is even, $\mu_1-t-1$ is odd, $B=0$, $s+2=\mu_2$.

$i=210+22\theta^{\mu_2-1}$, $j=111+221(-1)+22\theta^{t+1}C_1$, $ij=211$,

$ji=211(-1)+21\theta^{t+1}C_1=-ij+ij^{t+1}C_1(j)$, $i^2=21\theta^{\mu_2-1}=ij^{\mu_2-1}$.

41. $s+2<\mu_2$, $\mu_1>\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$, $B=\theta^{s+1-r}B_4$.

$i=210+22\theta^{s+1}+12\delta^{\mu_1-\mu_2+s+1-r}B_4$, $j=111+221+22\theta^{t+1}C_1=111+22\theta C$.

$ij^n=21n+12\delta^{\mu_1-t+n-1}C^nB_4+22\theta^{s+n+1}C^n$,

$ji=21\theta C+22\theta^{s+2}C+12\delta^{\mu_1-t}B_4=ijC(j)=ij+ij^{t+1}C_1(j)$.

$i^2=21\theta^{s+1}+22\theta^{\mu_1-t-1}B_4+11\delta^{\mu_1-t-1}B_4+12\delta^{\mu_1-t+s}B_4+22\theta^{2s+2}$

$=ij^{s+1}+j^{\mu_1-t-1}B_4$.

42. $s+2<\mu_2$, $\mu_1=\mu_2$, $c_0=1$, $t=\mu_2-s-2+r$, $B=\theta^{s+2-r}B_6$.

$i=210+22\theta^{s+1}+12\delta^{s+2-r}B_6$, $j=111+22\theta C$,

$ij^n=21n+22\theta^{s+n+1}C^n+12\delta^{s+n+2-r}B_6C^n$,

$ji=21\theta C+22\theta^{s+2}C+12\delta^{s+2-r}B_6=ijC(j)=ij+ij^{t+1}C_1(j)$,

$ij^2=21\theta^{s+1}+22\theta^{2s+2}+22\theta^{s+2-r}B_6+11\delta^{s+2-r}B_6+12\delta^{2s+3-r}B_6$

$=ij^{s+1}+j^{s+2-r}B_6(j)$.

| Type | $ji=$ | $i^2=$ | |
|---|---|---|---|
| 1 | $ij+ij^{t+1}C_1(j)$, | $ij^{\mu_2-1}+j^{\mu_1-t-1}B_3(j)$, | $\mu_1>\mu_2$, |
| 2 | $ij+ij^{t+1}C_1(j)$, | $ij^{\mu_2-1}+j^{\mu_1-t}B_5(j)$, | $\mu_1=\mu_2$, |
| 3 | $ij+ij^{t+1}C_1(j)$, | $ij^{s+1}+j^{\mu_1-t-1}B_4(j)$, | $\mu_1>\mu_2$, |
| 4 | $ij+ij^{t+1}C_1(j)$, | $ij^{s+1}+j^{\mu_2-t}B_6(j)$, | $\mu_1=\mu_2$, |
| 5 | $c_0ij+ij^{t+1}C_1(j)$, | $ij^{\mu_2-1}$, | $c_0^2\neq 1$, $\mu_1=\mu_2$, |
| 6 | $c_0ij+ij^{t+1}C_1(j)$, | $ij^{\mu_2-1}+j^{\mu_1-1}b$, | $c_0^2\neq 1$, $\mu_1>\mu_2$, |
| 7 | $c_0ij+ij^{t+1}C_1(j)$, | $ij^{\mu_2-1}+j^{\mu_1-1}b$, | $c_0^2\neq 1$, $c_0^{\mu_2-1}=1$, |
| 8 | $-ij+ij^{t+1}C_1(j)$, | $ij^{\mu_2-1}$, | $\mu_1-t$ is even, $\mu_2$ is odd, |
| 9 | $-ij+ij^{t+1}C_1(j)$, | $ij^{\mu_2-1}+j^{\mu_1-1}b$, | $\mu_1-t$ is odd, $\mu_2$ is odd, |
| 10 | $-ij+ij^{t+1}C_1(j)$, | $ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j)$, | $v\geq 2\mu_2-\mu_1$, $\mu_1-v$ is odd, |
| 11 | $-ij+ij^{t+1}C_1(j)$, | $ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j)$, | $v\geq 2\mu_2-\mu_1-1$, $\mu_1-v$ is even. |

*Case III$_2$.* $c_0=0$.

$C=\theta^tC_1(\theta)$, $0<t\leq\mu_2-2$. We go back to equations (A) which now become,

(1) $\theta^{t+2}A[1-C(\theta C)]=0$, for which we have either

1. $t=\mu_2-2$, $C=\theta^{\mu_2-2}c$, where $c$ is written for $c_{\mu_2-2}$, or
2. $A=\theta^{\mu_2-t-2}A_1$.

(2) $\delta^{\mu_1-\mu_2+1}B=0$. $\therefore B=\delta^{\mu_2-1}b$, where $b$ may be zero.

(3) $\delta^{\mu_1-\mu_2+1}B[A-C\cdot A(\theta C)]=0$. This is satisfied by the result of (2).

We now use the results of equations (1) and (2) in (4). For 1 $t=\mu_2-2$, $C=\theta^{\mu_2-2}c$, $B=\delta^{\mu_2-1}b$, equation (4) becomes

(4) $\theta^2A[A-\theta^{\mu_2-2}cA(\theta C)]+\theta^{\mu_1-1}b[1-C^{\mu_1-1}]=0$, $\theta^2A^2+\theta^{\mu_1-1}b=0$.

The three possible solutions of this are

11, $\mu_1>\mu_2$, $A=\theta^{m-1}A_2$ where $m=\mu_2/2$ if $\mu_2$ is even,

$$m=\frac{\mu_2-1}{2} \text{ if } \mu_2 \text{ is odd,}$$

12, $\mu_1=\mu_2=\mu$, $A=\theta^{\frac{\mu-2}{2}}A_3$, $B=0$, $\mu$ is even.

13, $\mu_1=\mu_2=\mu$, $A=\theta^{\frac{\mu-3}{2}}A_4$, $B=\delta^{\mu_2-1}b$, $a_{40}^2+b=0$, $\mu$ is odd.

For 2 $C=\theta^tC_1$, $B=\delta^{\mu_2-1}b$, $A=\theta^{\mu_2-t-2}A_1$, (4) becomes

(4) $\theta^{2\mu_2-2t-2}A_1[A_1-C^{\mu_2-t-1}A_1(\theta C)]+\theta^{\mu_1-1}b=0$,

$\theta^{2(\mu_2-t-1)}A_1^2-\theta^{(\mu_2-t-1)t+2\mu_2-2t-2}A_1C_1A_1(\theta C)+\theta^{\mu_1-1}b=0$.

This reduces to

$$\theta^{2(\mu_2-t-1)}A_1^2+\theta^{\mu_1-1}b=0,$$

because $(\mu_2-t-1)t+2\mu_2-2t-2\geq\mu_2$, *i. e.*, $(t+2)\mu_2-(t^2+3t+2)\geq\mu_2$, or $\mu_2-t-2\geq 0$, $t\leq\mu_2-2$, which is true for this case.

Now the possible solutions of

$$\theta^{2(\mu_2-t-1)}A_1^2+\theta^{\mu_1-1}b=0$$

are 21, $\mu_1>\mu_2$, which gives rise to

211, $2t\leq\mu_2-2$,

and 212, $2t>\mu_2-2$, $A_1=\theta^{t+1-m}A_2$, $m=\frac{\mu_2}{2}$ if $\mu_2$ is even,

$m=\frac{\mu_2+1}{2}$ if $\mu_2$ is odd,

and 22, $\mu_1=\mu_2=\mu$, which gives rise to the following:

221, $2t\leq\mu_2-2$, $b=0$,

222, $2t>\mu_2-2$, $A_1=\theta^{t+1-\frac{\mu}{2}}A_2$, $b=0$, $\mu$ is even,

223, $2t>\mu-2$, $A_1=\theta^{t+1-\frac{\mu-1}{2}}A_2'$, $a_{20}^2+b=0$, $\mu$ is odd.

Therefore we have the following eight types:

11. $\mu_1>\mu_2$, $A=\theta^{m-1}A_2$, $B=\theta^{\mu_2-1}b$, $C=\theta^{\mu_2-2}c$, $m=\frac{\mu_2}{2}$ if $\mu_2$ is even.

$m=\frac{\mu_2-1}{2}$ if $\mu_2$ is odd.

12. $\mu_1=\mu_2=\mu$ is even, $A=\theta^{\frac{\mu-2}{2}}A_3$, $B=0$, $C=\theta^{\mu-2}c$.

13. $\mu_1=\mu_2=\mu$ is odd, $A=\theta^{\frac{\mu-3}{2}}A_4$, $B=\theta^{\mu-1}b$, $C=\theta^{\mu-2}c$, $a_{40}^2+b=0$.

211. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-t-2}A_1$, $B=\delta^{\mu_2-1}b$, $C=\theta^tC_1$.

212. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-m-1}A_2$, $B=\delta^{\mu_2-1}b$, $C=\theta^tC_1$,

$m=\frac{\mu_2}{2}$ if $\mu_2$ is even, $m=\frac{\mu_2+1}{2}$ if $\mu_2$ is odd.

221. $\mu_1=\mu_2=\mu$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-t-2}A_1$, $B=0$, $C=\theta^tC_1$.

222. $\mu_1=\mu_2=\mu$ is even, $2t>\mu-2$, $A=\theta^{\frac{\mu}{2}-1}A_2$, $B=0$, $C=\theta^tC_1$.

223. $\mu_1=\mu_2=\mu$ is odd, $2t>\mu-2$, $A=\theta^{\frac{\mu-1}{2}-1}A_2'$, $B=\delta^{\mu-1}b$, $C=\theta^tC_1$, $a_{20}^2+b=0$.

These eight written out give the following:

11. $\mu_1>\mu_2$, $A=\theta^{m-1}A_2$, $B=\delta^{\mu_2-1}b$, $C=\theta^{\mu_2-2}C_1$, $m=\frac{\mu_2}{2}$ if $\mu_2$ is even,

$m=\frac{\mu_2-1}{2}$ if $\mu_2$ is odd.

$i=210+22\theta^mA_2+12\delta^{\mu_1-1}b$, $j=111+22\theta^{\mu_2-1}c$.

Hence $ij=211$, $ji=21\theta^{\mu_2-1}c=ij^{\mu_2-1}c$, $i^2=21\theta^mA_2+11\delta^{\mu_1-1}b=ij^mA_2(j)+j^{\mu_1-1}b$.

12. $\mu_1=\mu_2=\mu$ is even, $A=\theta^{\frac{\mu-2}{2}}A_3$, $B=0$, $C=\theta^{\mu-2}c$.

$i=210+22\theta^{\frac{\mu}{2}}A_3$, $j=111+22\theta^{\mu-1}c$, $ij=211$,

$ji=21\theta^{\mu-1}c=ij^{\mu-1}c$, $i^2=21\theta^{\frac{\mu}{2}}A_3=ij^{\frac{\mu}{2}}A_3(j)$.

13. $\mu_1=\mu_2=\mu$ is odd, $A=\theta^{\frac{\mu-3}{2}}A_4$, $B=\delta^{\mu_1-1}b$, $C=\theta^{\mu-2}c$, $a_{40}^2+b=0$.

$i=210+22\theta^{\frac{\mu-1}{2}}A_4+12\delta^{\mu-1}b$, $j=111+22\theta^{\mu-1}c$, $ij=211$,
$ji=21\theta^{\mu-1}c=ij^{\mu-1}c$,
$i^2=21\theta^{\frac{\mu-1}{2}}A_4+22\theta^{\mu-1}a_{40}^2+22\theta^{\mu-1}b+11\delta^{\mu-1}b=ij^{\frac{\mu-1}{2}}A_4(j)+j^{\mu-1}b$.

211. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-t-2}A_1$, $B=\delta^{\mu_2-1}b$, $C=\theta^tC_1$.

$i=210+22\theta^{\mu_2-t-1}A_1+12\delta^{\mu_1-1}b$, $j=111+22\theta^{t+1}C_1$, $ij=211$,
$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\mu_2-t-1}A_1+11\delta^{\mu_1-1}b=ij^{\mu_2-t-1}A_1(j)+j^{\mu_1-1}b$.

212. $\mu_1>\mu_2$, $2t\leq\mu_2-2$, $A=\theta^{\mu_2-m-1}A_2$, $B=\delta^{\mu_2-1}b$, $C=\theta^tC_1$,

$m=\frac{\mu_2}{2}$ if $\mu_2$ is even, $m=\frac{\mu_2+1}{2}$ if $\mu_2$ is odd.

$i=210+22\theta^{\mu_2-m}A_2+12\delta^{\mu_1-1}b$, $j=111+22\theta^{t+1}C_1$, $ij=211$,
$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\mu_2-m}A_2+11\delta^{\mu_1-1}b=ij^{\mu_2-m}A_2(j)+j^{\mu_1-1}b$.

221. $\mu_1=\mu_2=\mu$, $2t\leq\mu_1-2$, $A=\theta^{\mu-t-2}A_1$, $B=0$, $C=\theta^tC_1$.

$i=210+22\theta^{\mu-t-1}A_1$, $j=111+22\theta^{t+1}C_1$, $ij=211$,
$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\mu-t-1}A_1=ij^{\mu-t-1}A_1(j)$.

222. $\mu$ is even, $2t>\mu-2$, $A=\theta^{\frac{\mu}{2}-1}A_2$, $B=0$, $C=\theta^tC_1$.

$i=210+22\theta^{\frac{\mu}{2}}A_2$, $j=111+22\theta^{t+1}C_1$, $ij=211$,
$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$, $i^2=21\theta^{\frac{\mu}{2}}A_2=ij^{\frac{\mu}{2}}A_2(j)$.

223. $\mu$ is odd, $2t>\mu-2$, $A=\theta^{\frac{\mu-1}{2}-1}A_2'$, $B=\delta^{\mu-1}b$, $C=\theta^tC_1$, $a_{20}'^2+b=0$.

$i=210+22\theta^{\frac{\mu-1}{2}}A_2'+12\delta^{\mu-1}b$, $j=111+22\theta^{t+1}C_1$, $ij=211$,
$ji=21\theta^{t+1}C_1=ij^{t+1}C_1(j)$,
$i^2=21\theta^{\frac{\mu-1}{2}}A_2'+11\delta^{\mu-1}b+22\theta^{\mu-1}b+22\theta^{\mu-1}a_{20}'^2=ij^{\frac{\mu-1}{2}}A_2'(j)+j^{\mu-1}b$.

| Type | $ji=$ | $i^2=$ | |
|---|---|---|---|
| 1 | $ij^{\mu_2-1}c$, | $ij^mA_2(j)+j^{\mu_1-1}b$, | $\mu_1>\mu_2$, |
| 2 | $ij^{\mu-1}c$, | $ij^{\frac{\mu}{2}}A_3(j)$, | $\mu_1=\mu_2=\mu$ is even, |
| 3 | $ij^{\mu-1}c$, | $ij^{\frac{\mu-1}{2}}A_4(j)+bj^{\mu-1}$, | $\mu$ is odd, $b+a_{40}^2=0$, |
| 4 | $ij^{t+1}C_1(j)$, | $ij^{\mu_2-t-1}A_1(j)+bj^{\mu_1-1}$, | |
| 5 | $ij^{t+1}C_1(j)$, | $ij^{\mu_2-m}A_2(j)+bj^{\mu_1-1}$, | $m=\frac{\mu_2}{2}$ if $\mu_2$ is even, $m=\frac{\mu_2+1}{2}$ if $\mu_2$ is odd. |
| 6 | $ij^{t+1}C_1(j)$, | $ij^{\mu-t-1}A_1(j)$, | $2t\leq\mu-2$, |
| 7 | $ij^{t+1}C_1(j)$, | $ij^{\frac{\mu}{2}}A_2(j)$, | $2t>\mu-2$, $\mu$ is even, |
| 8 | $ij^{t+1}C_1(j)$, | $ij^{\frac{\mu-1}{2}}A_2'(j)+j^{\mu-1}b$, | $b+a_{20}'^2=0$, $2t>\mu-2$, $\mu$ is odd. |

VII. *Complete List of the Types Arranged According to the Form of the Product ji, with Sufficient Conditions to Distinguish Each.*

| Class | Type | $ji=$ | $i^2=$ | Conditions |
|---|---|---|---|---|
| I | 1 | $ij,$ | $i'^2=j^{\mu_1-\mu_2}B(j),$ | $E=0.$ |
| II | 2 | $ij+j^{\mu_2+1}E_1(j),$ | $i'^2=j^{\mu_2}B(j),$ | $\mu_1-\mu_2=\mu_2,\ E_1\neq 0,$ |
| | 3 | $ij+j^{\mu_2+\alpha}E(j),$ | $i'^2=j^{\mu_2+\alpha}B(j),$ | $\mu_1-\mu_2=\mu_2+\alpha,\ \alpha>0,\ E\neq 0,$ |
| | 4 | $ij+j^{\mu_1-\mu_2+v}E_3(j),$ | $i'^2=j^{\mu_1-\mu_2}B(j),$ | $\mu_1-\mu_2<\mu_2,\ v>2\mu_2-\mu_1,\ E_3\neq 0.$ |
| III | 5 | $ij+ij^{t+1}C_1(j),$ | $j^{\mu_1-t}B_2(j),$ | $\mu_1=\mu_2,$ |
| | 6 | $ij+ij^{t+1}C_1(j),$ | $j^{\mu_1-t-1}B_1(j),$ | $\mu_1>\mu_2,$ |
| | 7 | $ij+ij^{t+1}C_1(j),$ | $ij^{\mu_2-1}+j^{\mu_1-t-1}B_3(j),$ | $\mu_1>\mu_2,$ |
| | 8 | $ij+ij^{t+1}C_1(j),$ | $ij^{\mu_2-1}+j^{\mu_1-t}B_5(j),$ | $\mu_1=\mu_2,$ |
| | 9 | $ij+ij^{t+1}C_1(j),$ | $ij^{s+1}+j^{\mu_1-t-1}B_4(j),$ | $\mu_1>\mu_2,$ |
| | 10 | $ij+ij^{t+1}C_1(j),$ | $ij^{s+1}+j^{\mu_2-t}B_6(j),$ | $\mu_1=\mu_2.$ |
| IV | 11 | $c_0ij+ij^{t+1}C_1(j),$ | $0,$ | $c_0^2\neq 1,$ |
| | 12 | $c_0ij+ij^{t+1}C_1(j),$ | $j^{\mu_1-1}b,$ | $c_0^2\neq 1,\ b\neq 0,$ |
| | 13 | $c_0ij+ij^{t+1}C_1(j),$ | $ij^{\mu_2-1}+j^{\mu_1-1}b,$ | $c_0^2\neq 1,\ \mu_1>\mu_2,$ |
| | 14 | $c_0ij+ij^{t+1}C_1(j),$ | $ij^{\mu_2-1},$ | $c_0^2\neq 1,\ \mu_1=\mu_2,$ |
| | 15 | $c_0ij+ij^{t+1}C_1(j),$ | $ij^{\mu_2-1}+j^{\mu_1-1}b,$ | $c_0^2\neq 1,\ c_0^{\mu_2-1}=1.$ |
| V | 16 | $-ij+ij^{t+1}C_1(j),$ | $0,$ | |
| | 17 | $-ij+ij^{t+1}C_1(j),$ | $j^{\mu_1-1}b,$ | $b\neq 0,$ |
| | 18 | $-ij+ij^{t+1}C_1(j),$ | $ij^{\mu_2-1},$ | $\mu_1-t$ is even, $\mu_2$ is odd, |
| | 19 | $-ij+ij^{t+1}C_1(j),$ | $j^{\mu_1-t-1}B_1(j),$ | $B_1(j)\neq 0,$ |
| | 20 | $-ij+ij^{t+1}C_1(j),$ | $ij^{\mu_2-1}+j^{\mu_1-1}b,$ | $\mu_1-t$ is odd, $\mu_2$ is odd, |
| | 21 | $-ij+ij^{t+1}C_1(j),$ | $ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j),$ | $v\geq 2\mu_2-\mu_1,\ \mu_1-v$ is odd, |
| | 22 | $-ij+ij^{t+1}C_1(j),$ | $ij^{\mu_2-1}+j^{\mu_1-\mu_2+v}B_2(j),$ | $v\geq 2\mu_2-\mu_1-1,\ \mu_1-v$ is even. |
| VI | 23 | $ij^{\mu_2-1}c,$ | $ij^mA_2(j)+j^{\mu_1-1}b,$ | $\mu_1>\mu_2$ |
| | 24 | $ij^{\mu-1}c,$ | $ij^{\frac{\mu}{2}}A_3(j),$ | $\mu_1=\mu_2=\mu$ is even, |
| | 25 | $ij^{\mu-1}c,$ | $ij^{\frac{\mu-1}{2}}A_4(j)+bj^{\mu-1},$ | $b+a_{40}^2=0,\ \mu$ is odd, |
| | 26 | $ij^{t+1}C_1(j),$ | $ij^{\mu_2-t-1}A_1(j)+bj^{\mu_1-1},$ | |
| | 27 | $ij^{t+1}C_1(j),$ | $ij^{\mu_2-m}A_2(j)+bj^{\mu_1-1},$ | $m=\frac{\mu_2}{2}$ if $\mu_2$ is even, $m=\frac{\mu_2+1}{2}$ if $\mu_2$ is odd, |
| | 28 | $ij^{t+1}C_1(j),$ | $ij^{\mu-t-1}A_1(j),$ | $2t\leq\mu-2,$ |
| | 29 | $ij^{t+1}C_1(j),$ | $ij^{\frac{\mu}{2}}A_2(j),$ | $2t>\mu-2,\ \mu$ is even, |
| | 30 | $ij^{t+1}C_1(j),$ | $ij^{\frac{\mu-1}{2}}A_2'(j)+bj^{\mu-1},$ | $2t>\mu-2,\ \mu$ is odd, $b+a_{10}'^2=0.$ |

These types when arranged according to the form of the product $ji$ fall into six classes.

Class I, made up of type 1, is the only commutative type of the entire set.

Class II, types 2–4, have for the product $ji$ the term $ij$ and powers of $j$.

Class III, types 5–10, have $ji=ij+ij^{t+1}C_1(j)$ where $t>0$ and $C_1\neq 0$.

Class IV, types 11–15, differ from those of Class III in that

$$ji=c_0ij+ij^{t+1}C_1(j) \text{ where } c_0^2\neq 1.$$

Class V, types 16–22, are essentially different from those of Classes III and IV since $ji=-ij+ij^{t+1}C_1(j)$.

Class VI, types 23–30, have the product $ji$ starting with at least $ij^2$.

We need now to see that the types of each class are essentially distinct. The conditions given, having arisen in the determination of these types, are sufficient to show this. For example, we might think that type 18 and type 20 would be the same if in type 20 $b=0$. Although this would make the products $ji$ and $i^2$ have the same expressions for the two types, they would still be distinct, for in 18 $\mu_1-t$ is odd and in 20 it is even. Types 13 and 15 have the same form for $ji$ and $i^2$, but they are essentially distinct because in 13 $\mu_1>\mu_2$, which is not necessarily the case in 15, and furthermore in 15 $c_0^{\mu_2-1}=1$, but the only limitation on $c_0$ in 13 is $c_0^2\neq 1$. Similarly it can be shown that the thirty types are all distinct.